New simple, cheap and efficient error estimator for adaptive fracture and damage analysis (in 2D)

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New simple, cheap and efficient error estimator for adaptive fracture and damage analysis (in 2D)

- Background: what is error estimator (EE)?
- Example of EE, how it works
- Motivation/Main result
- Applications: fracture & damage analysis
Problem strong formulation:

\[
\begin{align*}
-\text{div } \sigma(u) &= f \quad \text{in} \quad \Omega \\
\sigma &= C : \varepsilon(u) \quad \text{in} \quad \Omega \\
\varepsilon &= \nabla^{\text{sym}} u \quad \text{in} \quad \Omega \\
\hline
u &= 0 \quad \text{on} \quad \Gamma_D \\
\sigma(u) \cdot n &= 0 \quad \text{on} \quad \Gamma_{N,0} \\
\sigma(u) \cdot n &= \bar{t} \quad \text{on} \quad \Gamma_{N,1} \\
\sigma(u) \cdot n^\pm &= 0 \quad \text{on} \quad \Gamma_c^\pm
\end{align*}
\]
Background
What is error estimator?

\[ u - u_h = e \]

discr. error

exact solution (unknown)

FE solution (available)

Problem strong formulation:

\[
\begin{aligned}
- \text{div} \, \sigma(u) &= f \quad \text{in} \quad \Omega \\
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\end{aligned}
\]
Background

What is error estimator?

\[ \| \mathbf{u} - \mathbf{u}_h \|_\Omega = \| \mathbf{e} \|_\Omega \]

discr. error

exact solution (unknown)

FE solution (available)

Problem strong formulation:

\[
\begin{align*}
- \text{div } \sigma(u) &= f & \text{in } & \Omega \\
\sigma &= \mathcal{C} : \varepsilon(u) & \text{in } & \Omega \\
\varepsilon &= \nabla^{\text{sym}} u & \text{in } & \Omega \\
\mathbf{u} &= 0 & \text{on } & \Gamma_D \\
\sigma(u) \cdot \mathbf{n} &= 0 & \text{on } & \Gamma_{N,0} \\
\sigma(u) \cdot \mathbf{n} &= \mathbf{t} & \text{on } & \Gamma_{N,1} \\
\sigma(u) \cdot \mathbf{n}^\pm &= 0 & \text{on } & \Gamma_c^\pm
\end{align*}
\]
Background
What is error estimator?

I. To assess the **accuracy** of your FE solution by computing the $UB$

\[ \|u - u_h\|_\Omega = \|e\|_\Omega \leq UB \]

Problem strong formulation:
\[
\begin{aligned}
\text{div } \sigma(u) &= f \quad \text{in } \Omega \\
\sigma &= C : \varepsilon(u) \quad \text{in } \Omega \\
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\end{aligned}
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\]
Background
What is error estimator?

discr. error

$$\|u - u_h\|_\Omega = \|e\|_\Omega \leq UB$$

I. To assess the **accuracy** of your FE solution by computing the **UB**

II. To trigger a refinement strategy (e.g. **mesh refinements**) => better **accuracy**

Problem strong formulation:

$$\begin{aligned}
-\text{div} \sigma(u) &= f & \text{in} & \Omega \\
\sigma &= C : \varepsilon(u) & \text{in} & \Omega \\
\varepsilon &= \nabla^{\text{sym}} u & \text{in} & \Omega \\
\Gamma_D & : u = 0 & \text{on} & \Gamma_D \\
\Gamma_{N,0} & : \sigma(u) \cdot n = 0 & \text{on} & \Gamma_{N,0} \\
\Gamma_{N,1} & : \sigma(u) \cdot n = \bar{t} & \text{on} & \Gamma_{N,1} \\
\Gamma_{c} & : \sigma(u) \cdot n^\pm = 0 & \text{on} & \Gamma_{c} 
\end{aligned}$$
**Problem strong formulation:**

\[
\begin{align*}
\text{div } \sigma(u) &= f \quad \text{in } \Omega \\
\sigma &= \mathcal{C} : \varepsilon(u) \quad \text{in } \Omega \\
\varepsilon &= \nabla^{\text{sym}} u \quad \text{in } \Omega \\
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\sigma(u) \cdot n^\pm &= 0 \quad \text{on } \Gamma_{c}^\pm
\end{align*}
\]

I. To assess the **accuracy** of your FE solution by computing the UB

II. To trigger a refinement strategy (e.g. **mesh refinements**) => better accuracy
**Background**

What is error estimator?

\[
\|e\|_\Omega \leq UB = \left(\sum_T \eta_T^2\right)^{1/2}
\]

Computable Upper Bound

Local error indicators

**Problem strong formulation:**

\[
\begin{align*}
-\text{div} \, \sigma(u) &= f \quad \text{in} \quad \Omega \\
\sigma &= \mathcal{C} : \varepsilon(u) \quad \text{in} \quad \Omega \\
\varepsilon &= \nabla^{\text{sym}} u \quad \text{in} \quad \Omega \\
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\sigma(u) \cdot n = \bar{t} \quad \text{on} \quad \Gamma_{N,1} \\
\sigma(u) \cdot n^\pm &= 0 \quad \text{on} \quad \Gamma_{\pm}
\end{align*}
\]
Example

Available classical Babuška-Miller estimator:

$$\|e\|_\Omega \leq \frac{C}{\sqrt{2} \mu} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}$$

$$\eta_{T,\text{Class}} := h_T \| f + \text{div} \sigma(u^h) \|_{L^2(\Omega)} + h_T^{1/2} \sum_{\ell=1}^3 \| \langle \sigma(u^h) \cdot n \rangle_{E_\ell} \|_{L^2(E_\ell)}$$

Problem strong formulation:

$$\begin{cases}
- \text{div} \sigma(u) = f & \text{in} \quad \Omega \\
\sigma = C : \varepsilon(u) & \text{in} \quad \Omega \\
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 u = 0 & \text{on} \quad \Gamma_D \\
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\end{cases}$$
Example

Available classical Babuška-Miller estimator:

\[ \|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: U_{B_{\text{Class}}} \]

\[ \eta_{T,\text{Class}} := h_T \|f + \text{div}\sigma(u^h)\|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^{3} \left\| \langle \sigma(u^h) \cdot n \rangle_{E_\ell} \right\|_{L^2(E_\ell)} \]

Derived straightforwardly by using the theory [1], [2], [3]


Example

Available classical Babuška-Miller estimator:

$$\|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} = UB_{\text{Class}}$$

$$\eta_{T,\text{Class}} := h_T \left\| f + \text{div}\sigma(u^h) \right\|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^3 \left\| \sigma(u^h) \cdot n \right\|_{L^2(E_\ell)}$$

- **diam(T)**: strong form of the interior element residual
- **tractions jumps at the element boundaries**
Available classical Babuška-Miller estimator:

$$\|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}$$

$$\eta_{T,\text{Class}} := h_T \left\| \begin{array}{c} f + \text{div } \sigma(u^h) \\ \text{div } \sigma(u^h) \end{array} \right\|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^3 \left\| \begin{array}{c} \sigma(u^h) \cdot n \\ \sigma(u^h) \cdot n \end{array} \right\|_{L^2(E_\ell)}$$

- **Example**

  a) **Explicit** (all data involved is explicitly available) \(\Rightarrow\) **simple** and **cheap**
Available classical Babuška-Miller estimator:

\[ \| e \| _{\Omega} \leq \frac{C}{\sqrt{2\mu}} \left( \sum_{T} \eta_{T,\text{Class}}^{2} \right)^{1/2} =: UB_{\text{Class}} \]

\[ \eta_{T,\text{Class}} := h_T \| f + \text{div}\sigma(u^h) \|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^{3} \left\| \left( \sigma(u^h) \cdot n \right) \right\|_{E_\ell} \]

Example

a) **Explicit** (all data involved is explicitly available) => **simple** and **cheap**

b) Basis for **adaptive** mesh refinements => you refine, where it is **appropriate**
**Example**

Plane strain

\[ E = 77.1 \text{ GPa} \]
\[ \nu = 0.33 \]
(aluminum 7075-T6)

Displacement controlled loading with
\[ u_0 = (0.156 \cdot 10^{-5}) \text{ mm} \]

*P1-triangular FEM*
Example

- Loading
- Crack
- Initial coarse mesh (2854 DOF)
- Deformed mesh (magn. $1.5 \cdot 10^5$)
Example

\[ \eta_{T,\text{Class}} := h_T \| f + \text{div} \sigma(u^h) \|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^3 \| \langle \sigma(u^h) \cdot n \rangle_{E_\ell} \|_{L^2(E_\ell)} \]

Initial coarse mesh (2854 DOF)

Distribution of local errors,

\[ \log_{10}(\eta_{T,\text{Class}}) \]
Example

Initial coarse mesh (2854 DOF)  Adaptive step 1 (5078 DOF)
Example

\[ \log_{10}(\eta_{T,\text{Class}}) \]
Different types of (strong/weak) singularities:

- Crack tip (limiting case of a re-entrant corner)
- The points, where the BC change
- Concave parts of the boundary

are naturally captured by mesh adaptivity
Example

Adaptive step 1
(5078 DOF)

Adaptive step 2
(9682 DOF)

e tc.
Example

Adaptive step 1
(5078 DOF)

Adaptive step 2
(9682 DOF)

etc.

(until when is to refine ?)
Example

Available classical Babuška-Miller estimator:

\[
\|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} = UB_{\text{Class}}
\]

\[
\eta_{T,\text{Class}} := h_T \| f + \text{div} \sigma(u^h) \|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^3 \left\| \left( \sigma(u^h) \cdot n \right)_{E_\ell} \right\|_{L^2(E_\ell)}
\]

Pros:

a) Explicit, simple and cheap
b) Basis for adaptive mesh refinements
Available **classical Babuška-Miller** estimator:

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\| e \|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}
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\]

**Pros:**

a) **Explicit, simple** and **cheap**

b) Basis for **adaptive** mesh refinements

**Cons:**

Multiplicative constant **C** is **not known**
Available classical Babuška-Miller estimator:

\[
\| e \|_\Omega \leq \frac{C}{\sqrt{2 \mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}
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\]

**Pros:**

a) Explicit, simple and cheap  
b) Basis for adaptive mesh refinements

**Cons:**

Multiplicative constant $C$ is not known => * $UB$ is not available (how accurate are you?)
Example

Available classical Babuška-Miller estimator:

\[ \| e \|_\Omega \leq \frac{C}{\sqrt{2 \mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}} \]

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Pros:

a) **Explicit, simple and cheap**
b) Basis for **adaptive** mesh refinements

Cons:

Multiplicative constant \( C \) is **not known** =>

* \( UB \) is not available (how accurate are you?)
* no stopping criterion for adaptivity
Example

Available classical Babuška-Miller estimator:

\[ \| e \|_{\Omega} \leq \frac{C}{\sqrt{2\mu}} \left( \sum_{T} \eta_{T,Class}^{2} \right)^{1/2} =: UB_{\text{Class}} \]

\[ \eta_{T,Class} := h_{T} \| f + \text{div}\sigma(u^{h}) \|_{L^{2}(T)} + h_{T}^{1/2} \sum_{\ell=1}^{3} \left\| \langle \sigma(u^{h}) \cdot n \rangle_{E_{\ell}} \right\|_{L^{2}(E_{\ell})} \]

**Pros:**

a) Explicit, simple and cheap
b) Basis for adaptive mesh refinements

**Cons:**

Multiplicative constant \( C \) is not known \( \Rightarrow \) * UB is not available (how accurate are you?)
* no stopping criterion for adaptivity

By no means \( C \) can be found through the actual derivation procedure
Example

What if to set \( C=1 \)?

\[
\| e \|_\Omega \leq \frac{C}{\sqrt{2 \mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}
\]
Example

What if to set $C = 1$?

\[ \|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_{T} \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}} \]
Example

What if to set $C=1$?

$$
\|e\|_{\Omega} \leq \frac{C}{\sqrt{2\mu}} \left( \sum_{T} \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}
$$

**NB:** reference ("exact") error $\|e\|_{\Omega}$ obtained through computing an overkill solution.
Example

What if to set $C=1$?

$$\| e \|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_T^{2,\text{Class}} \right)^{1/2} =: UB_{\text{Class}}$$

Example

What if to set $C=1$?

$$\| e \|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_T^{2,\text{Class}} \right)^{1/2} =: UB_{\text{Class}}$$

$C \equiv 1$
Example

What if to set $C=1$?

\[ \|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}} \]

Is not allowed; if done – practically useless estimator
Motivation / Main result

Available classical Babuška-Miller estimator:

\[ \|\mathbf{e}\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}} \]

\[ \eta_{T,\text{Class}} := h_T \left\| f + \text{div}\sigma(u^h) \right\|_{L^2(T)} + h_T^{1/2} \sum_{\ell=1}^3 \left\| \langle \sigma(u^h) \cdot \mathbf{n} \rangle_{E_{\ell}} \right\|_{L^2(E_{\ell})} \]

Pros:

a) Explicit, simple and cheap
b) Basis for adaptive mesh refinements

Cons:

Multiplicative constant \( C \) is not known
Motivation / Main result

Available classical Babuška-Miller estimator:

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\|e\|_{\Omega} \leq \frac{C}{\sqrt{2\mu}} \left( \sum_{T} \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}
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\]

**Pros:**

a) Explicit, simple and cheap
b) Basis for adaptive mesh refinements

**Cons:**

Multiplicative constant \(C\) is not known

My motivation: Pros ! Cons
Motivation / Main result

Available classical Babuška-Miller estimator:

\[ \| \mathbf{e} \|_{\Omega} \leq \frac{C}{\sqrt{2 \mu}} \left( \sum_{T} \eta_{T,Class}^{2} \right)^{1/2} =: UB_{Class} \]

\[ \eta_{T,Class} := h_{T} \left\| \mathbf{f} + \text{div}\sigma(\mathbf{u}^{h}) \right\|_{L^{2}(T)} + h_{T}^{1/2} \sum_{\ell=1}^{3} \left\| \left\langle \sigma(\mathbf{u}^{h}) \cdot \mathbf{n} \right\rangle_{E_{\ell}} \right\|_{L^{2}(E_{\ell})} \]
Motivation / Main result

Available classical Babuška-Miller estimator:

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\]

We propose the constant-free estimator:

\[
\|e\|_\Omega \leq \frac{\widetilde{c}_p \widetilde{c}_K}{\sqrt{2\mu + \widetilde{c}_{SE} \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}}
\]
Motivation / Main result

Available classical Babuška-Miller estimator:

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\]

We propose the constant-free estimator:

\[
\|e\|_\Omega \leq \frac{\tilde{c}_p \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE} \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}}
\]

\[
\tilde{c}_p = \frac{4(\sqrt{17} - 1)^{1/2}}{(7 + \sqrt{17})(3 + \sqrt{17})^{1/2}} \quad \tilde{c}_K = 2 \left( \frac{\pi}{3\pi + 2} \right)^{1/2} \quad \tilde{c}_{SE} = \frac{2\pi + 4}{3\pi + 2}
\]
Motivation / Main result

Available classical Babuška-Miller estimator:

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We propose the constant-free estimator:

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\|\mathbf{e}\|_{\Omega} \leq \frac{\tilde{c}_p \tilde{c}_K}{\sqrt{2 \mu + \tilde{c}_SE \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} := UB_{\text{New}}
\]

\[
\eta_{T,\text{New}} := h_T \| \mathbf{f} + \text{div} \sigma (\mathbf{u}^h) \|_{L^2(T)} + \frac{h_T}{|T|^{1/2}} \sum_{\ell=1}^3 \left\| E_\ell^{1/2} \right\|_{L^2(E_\ell)} \left\| \langle \sigma (\mathbf{u}^h) \cdot \mathbf{n} \rangle_{E_\ell} \right\|_{L^2(E_\ell)}
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We propose the constant-free estimator:

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Motivation / Main result

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\| e \|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}} \right)^{1/2} =: UB_{\text{Class}}
\]
Motivation / Main result

\[ \| \mathbf{e} \|_{\Omega} \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta^2_{T,\text{Class}} \right)^{1/2} =: UB_{\text{Class}} \]

\[ \| \mathbf{e} \|_{\Omega} \leq \frac{\tilde{c}_p \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE}\lambda}} \left( \sum_T \eta^2_{T,\text{New}} \right)^{1/2} =: UB_{\text{New}} \]

\[ \text{Error} \]

\[ \text{number DOF} \]

\[ UB_{\text{Class}}, \quad C \equiv 1 \]

\[ UB_{\text{New}} \]
Motivation / Main result

\[ \| e \|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}} \]

\[ \| e \|_\Omega \leq \frac{\tilde{c}_p \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE}\lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}} \]

**New Upper Bound** is guaranteed and accurate
Motivation / Main result

\[
\| e \|_{\Omega} \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}
\]

\[
\| e \|_{\Omega} \leq \frac{c_p c_K}{\sqrt{2\mu + \tilde{c}_{SE} \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}}
\]

Corresponding effectivity index is close to 1!
Motivation / Main result

The above geometries were used for the primal testing of

\[ \| e \|_\Omega \leq \frac{\tilde{c}_p \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE} \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}} \]

plane stress
\[ E = 65000 \]
\[ \nu = 0.29 \]

P1-triangular FEM
1. In all examples considered \( \theta_{\text{New}} := \frac{UB_{\text{New}}}{\|e\|_{\Omega}} < 2 \) (namely, 1.3-1.6) both for the \textit{uniform} and \textit{adaptive} mesh refinement strategies.

\[ E = 65000 \]
\[ \nu = 0.29 \]
1. In all examples considered, \( \theta_{\text{New}} := \frac{UB_{\text{New}}}{\| e \|_\Omega} < 2 \) (namely, 1.3-1.6) both for the uniform and adaptive mesh refinement strategies.

2. In the letter case, for \( p=1 \) the optimal convergence rate (1 and 0.5 in terms of \( h \) and \( N \), respectively) is restored (smt. doubled).

\[
\| e \|_\Omega \leq C h^{\min(p, \lambda)} \quad \| e \|_\Omega \leq \tilde{C} N^{\frac{1}{\min(p, \lambda)}}
\]
Motivation / Main result

Stopping criterion for mesh refinements (adaptive/uniform) is now “well-defined”:

\[
\frac{\|e\|^2_\Omega}{a(u,u)} \leq \text{TOL}
\]
Motivation / Main result

Stopping criterion for mesh refinements (adaptive/uniform) is now “well-defined”:

\[
\frac{\|e\|_\Omega^2}{a(u,u)} \leq \text{TOL}
\]

Along with

\[
\|e\|_\Omega \leq UB_{\text{New}}
\]

\[
a(u,u) = \|e\|_\Omega^2 + a(u^h, u^h)
\]
Motivation / Main result

Stopping criterion for mesh refinements (adaptive/uniform) is now “well-defined”:

\[
\frac{\|e\|^2_\Omega}{a(u,u)} \leq \text{TOL}
\]

Along with

\[
a(u,u) = \|e\|^2_\Omega + a(u^h,u^h) \leq UB_{New}
\]

\[
\frac{\|e\|^2_\Omega}{a(u,u)} \leq \frac{1}{1 + \frac{a(u^h,u^h)}{UB_{New}^2}}
\]
Motivation / Main result

Stopping criterion for mesh refinements (adaptive/uniform) is now “well-defined”:

$$\frac{\|e\|^2_\Omega}{a(u,u)} \leq \text{TOL}$$

Along with

$$\|e\|^2_\Omega \leq UB_{New}$$

$$a(u,u) = \|e\|^2_\Omega + a(u^h,u^h)$$

$$\frac{\|e\|^2_\Omega}{a(u,u)} \leq \frac{1}{1 + \frac{a(u^h,u^h)}{UB_{New}^2}} \leq \text{TOL}$$
Application I:
Error-controlled crack propagation (in 2D)

Experimental setup [1]:

\[ E = 77.1 \text{ GPa} \]
\[ v = 0.33 \]
(aluminum 7075-T6)

Application I: Error-controlled crack propagation (in 2D)

Experimental setup [1]:

\[ E = 77.1 \text{ GPa} \]
\[ \nu = 0.33 \]
(aluminum 7075-T6)

Application I: Error-controlled crack propagation (in 2D)

Experimental setup [1]:

Our setting:

\[ E = 77.1 \text{ GPa} \]
\[ \nu = 0.33 \]
(aluminum 7075-T6)

Application I: Error-controlled crack propagation (in 2D)

Experimental setup [1]:

Our setting:

13 propagation steps, each with
- displ. $u_0 = (0, 1.56 \cdot 10^{-5}) \text{ mm}$
- crack increment $\Delta a = 2.5 \text{ mm}$

$E = 77.1 \text{ GPa}$
$\nu = 0.33$
(aluminum 7075-T6)

Application I:
Error-controlled crack propagation (in 2D)

Experimental setup [1]:

Our setting:

- the domain expression for the $J$-integral:

$$J(u; \theta) = - \int_{\Omega J} \nabla (q\bar{n}) : (W_s(u) I - \nabla^T u \cdot \sigma(u)) \, dx \, dy,$$

13 propagation steps, each with
- displ. $u_0 = (0, 1.56 \cdot 10^{-5}) \, mm$
- crack increment $\Delta a = 2.5 \, mm$

$E = 77.1 \, GPa$
$\nu = 0.33$
(aluminum 7075-T6)

plane-strain

Experimental setup [1]:

Our setting:

Our result [2]:

Application I:
Error-controlled crack propagation (in 2D)

$E = 77.1 \text{ GPa}$

$\nu = 0.33$

(aluminum 7075-T6)

Application I: Error-controlled crack propagation (in 2D)

Prop. Step 0 (pre-existing crack)

Initial mesh
(2854 DOF)

Adapt. Step 3
(20390 DOF)

TOL = 4%

Propagation angle
$\theta_{\text{prop}} = 1.525^\circ$
for the next prop. step

Propagating crack opening
Application I: Error-controlled crack propagation (in 2D)

Initial mesh
(2854 DOF)

Adapt. Step 3
(20714 DOF)

TOL = 4%

Propagation angle
θ_{prop} = 0.875°
for the next prop. step

Propagating crack opening
Application I: Error-controlled crack propagation (in 2D)

Prop. Step 2

TOL = 4%

Initial mesh (2850 DOF)

Adapt. Step 3 (21294 DOF)

Propagation angle

$\theta_{\text{Prop}} = -0.05^\circ$

for the next prop. step

Propagating crack opening
Application I: Error-controlled crack propagation (in 2D)

Prop. Step 3

Initial mesh (2830 DOF)

Adapt. Step 3 (20736 DOF)

TOL = 4%

Propagation angle

\[ \theta_{\text{prop}} = -1.4^\circ \]

for the next prop. step

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

Initial mesh
(2862 DOF)

Adapt. Step 3
(20858 DOF)

TOL = 4%

Propagation angle
$\theta_{\text{prop}} = -3.05^\circ$
for the next prop. step

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 5

Initial mesh
(2834 DOF)

Adapt. Step 3
(21216 DOF)

TOL = 4%

Propagation angle
\( \theta_{\text{prop}} = -5.34^\circ \)
for the next prop. step

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 6

TOL = 4%

Initial mesh
(2844 DOF)

Adapt. Step 3
(20732 DOF)

Propagation angle
\[ \theta_{\text{Prop}} = -8.175^\circ \]
for the next prop. step

Propagating crack opening
Application I: Error-controlled crack propagation (in 2D)

Prop. Step 7

Initial mesh (2846 DOF)  Adapt. Step 3 (21900 DOF)

TOL = 4%

Propagation angle $\theta_{\text{Prop}} = -11.81^\circ$

for the next prop. step

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

**Prop. Step 8**

- **Initial mesh** (2856 DOF)
- **Adapt. Step 3** (21196 DOF)

TOL $= 4\%$

Propagation angle $\theta_{\text{Prop}} = -16.425^\circ$

for the next prop. step

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

**Prop. Step 9**

- **Initial mesh** (2846 DOF)
- **Adapt. Step 3** (21928 DOF)

Propagation angle
\[ \theta_{\text{Prop}} = -22.46^\circ \]
for the next prop. step

TOL = 4%
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 10

Initial mesh (2842 DOF)

Adapt. Step 3 (21648 DOF)

Propagation angle
\[ \theta_{\text{Prop}} = -30.675^\circ \]
for the next prop. step

TOL = 4%

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 11

Initial mesh
(2870 DOF)

Adapt. Step 3
(21868 DOF)

TOL = 4%

Propagation angle
\[ \theta_{\text{Prop}} = -41.85^\circ \]

for the next prop. step

Propagating crack opening
Application I: Error-controlled crack propagation (in 2D)

Prop. Step 12

Initial mesh (2858 DOF)

Adapt. Step 3 (21922 DOF)

TOL = 4%

Propagation angle

$\theta_{\text{Prop}} = -54.82^\circ$

for the next prop. step

Propagating crack opening
Application I:
Error-controlled crack propagation (in 2D)

Initial mesh
(2868 DOF)

Adapt. Step 3
(24908 DOF)

TOL = 4%

Final failure

Prop. Step 13
(we stop here)
Application I:
Error-controlled crack propagation (in 2D)

was modeled
Application I: Error-controlled crack propagation (in 2D)

Note,

- **low-order elements** were used (the costs are really affordable)
- a **standard** (not purpose-oriented) **comp. code** [1] was used
- a progressing crack was modeled as a **wedge** (not a slit, as e.g. in XFEM/GFEM)
- **TOL:=4%** for adaptive remeshing, and crack **increment** were relatively **large**.

Yet, the results are accurate enough, what we mainly attribute to mesh adaptivity, driven by the new error estimator proposed.

---

Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:

Imagine a diagram showing a concrete beam with labels indicating distances and forces applied at various points. The beam has labels such as 20 mm, 180 mm, and 40 mm, along with force indications like $\frac{1}{10} P$ and $\frac{1}{11} P$. The beam is supported at the ends and shows failure modes at different sections.

Also in [2] for the “small” specimen.


Application II: Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:

We are interested in accurate modeling of this particular case:

the two non-symmetric cracks nucleate and evolve
(the left one is seemingly “counter intuitive”)
Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:

We adopt the comput. model from [2]:

\[
\sigma = (1 - D(\bar{\varepsilon}_{eq})) \mathbb{C} : \varepsilon \quad \text{in } \Omega \\
\begin{cases}
-c \nabla^2 \bar{\varepsilon}_{eq} + \bar{\varepsilon}_{eq} = \varepsilon_{eq}(\varepsilon) \quad \text{in } \Omega \\
\nabla \bar{\varepsilon}_{eq} \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega
\end{cases}
\]

- specific definition of equivalent strain

\[\varepsilon_{eq} := f(k, J_1(\varepsilon), J_2(\varepsilon))\]

- calibrated: \(k, c\) and \(\kappa_0, \alpha, \beta\) in \(D\)

We first adapt mesh on the pre-damaged stage...

Application II: Quasi-error-controlled damage analysis for a failure of SEN concrete beam
Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

We first adapt mesh on the pre-damaged stage...

\[
\sigma = (1 - D(\bar{\varepsilon}_{eq})) \mathbf{C} : \varepsilon \quad \text{in} \quad \Omega
\]

\[
\begin{align*}
-\varepsilon \nabla^2 \bar{\varepsilon}_{eq} + \bar{\varepsilon}_{eq} & = \varepsilon_{eq}(\varepsilon) \quad \text{in} \quad \Omega \\
\nabla \bar{\varepsilon}_{eq} \cdot \mathbf{n} & = 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

...when the above system is naturally decoupled.
Application II: Quasi-error-controlled damage analysis for a failure of SEN concrete beam

We first adapt mesh on the pre-damaged stage...

\[ \sigma = (1 - D(\bar{\epsilon}_{eq})) C : \epsilon \quad \text{in} \quad \Omega \]
\[ -c \nabla^2 \bar{\epsilon}_{eq} + \bar{\epsilon}_{eq} = \bar{\epsilon}_{eq}(\epsilon) \quad \text{in} \quad \Omega \]
\[ \nabla \bar{\epsilon}_{eq} \cdot n = 0 \quad \text{on} \quad \partial \Omega \]

...when the above system is naturally decoupled.

Hence, we can use our error estimator

\[ \| e \|_\Omega \leq \frac{\tilde{C}_p \tilde{C}_K}{\sqrt{2\mu + \tilde{C}_SE\tilde{\lambda}}} \left( \sum_T \eta_{T,New}^2 \right)^{1/2} =: UB_{New} \quad \text{TOL} = 1 \% \]
Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

We first adapt mesh on the pre-damaged stage...

... and then compute the damage $D$ evolution

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</tr>
</tbody>
</table>
Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:

Damage state at a certain loading [2]:

Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

**Adapted** mesh for damage evolution computations

26986 DOF

**Uniform** mesh

20684 DOF

Damage state for the same loading
Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Adapted mesh
for damage evolution computations

Uniform mesh
26986 DOF

20684 DOF

Damage state for the same loading

Transformation of a damage zone into an equivalent crack, [1]

Application II:
Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Adapted mesh
for damage evolution computations

Uniform mesh

Damage state for the same loading

Transformation of a damage zone into an equivalent crack, [1]

Application III: 
Micro-crack initiation in a ceramic specimen

Magnesium-stabilized Zirconia dioxide (Mg-ZrO$_2$) ceramic beam specimen of size 25x2x2 (dim. in mm) with a micro-notch of length $\sim$330 $\mu$m under 3-point bending test

Micro-crack, stemming from the notch tip, shortly before rapture
Application III:
Micro-crack initiation in a ceramic specimen (first attempt)

Initiation and development of a micro-crack
Application III:
Micro-crack initiation in a ceramic specimen (first attempt)

Micro-crack nucleates inside of the specimen, namely, on the grain boundary and develops along it.
Application III:
Micro-crack initiation in a ceramic specimen (first attempt)

Micro-crack nucleates inside of the specimen, namely, on the grain boundary and develops along it.

We will model this phenomena by using the damage model.
Application III: Micro-crack initiation in a ceramic specimen (first attempt)

Our setup:

\[ u = 0 \quad \text{width} = 1.96 \text{ mm} \]

\[ u_y = 0, \ t_x = 0 \]

25 mm

1.57 mm
Application III: Micro-crack initiation in a ceramic specimen (first attempt)

Our setup:

\[ u = 0 \]
\[ u_y = 0, t_x = 0 \]

width = 1.96 mm
25 mm
1.57 mm

Let’s consider only one grain (an inclusion in blue), perfectly bonded with the outer material (in red) and such that \( E_1 \neq E_2 \)

\[ E_1 = 200 \text{ GPa} \]
\[ \nu_1 = 0.3 \]

\[ E_2 = 250 \text{ GPa} \]
\[ \nu_2 = 0.3 \]

diam = 40 \mu m
Application III:
Micro-crack initiation in a ceramic specimen (first attempt)

Our setup:

\[ u = 0 \quad u_y = 0, \quad t_x = 0 \]

width = 1.96 mm

width = 1.57 mm

\[ E_1 = 200 \text{ GPa} \]
\[ \nu_1 = 0.3 \]

\[ E_2 = 250 \text{ GPa} \]
\[ \nu_2 = 0.3 \]

diam = 40 \mu m

Micro-crack nucleates \textit{inside} of the specimen, namely, on the \textbf{grain boundary} and develops along it.
Application III: Micro-crack initiation in a ceramic specimen (first attempt)

Our setup:

\[ u = 0 \quad u_y = 0, \; t_x = 0 \]

width = 1.96 mm

Initial coarse mesh

Adapted mesh (before initiation of damage)
Application III:
Micro-crack initiation in a ceramic specimen (first attempt)

Evolution of damage $D$ (before the peak load)
Application III:
Micro-crack initiation in a ceramic specimen (first attempt)

Evolution of damage $D$ (before the peak load)

1

2

3

4

5

6

$E_1 = 200$ GPa
$\nu_1 = 0.3$

$E_2 = 250$ GPa
$\nu_2 = 0.3$

diam $= 40$ $\mu$m

330 $\mu$m

100 $\mu$m

100 $\mu$m
Application III: Micro-crack initiation in a ceramic specimen (first attempt)

Evolution of damage $D$ (before the peak load)

$D_{\text{max}}$ is attained on the grain boundary

$E_1 = 200 \text{ GPa}$
$\nu_1 = 0.3$

$E_2 = 250 \text{ GPa}$
$\nu_2 = 0.3$

Diam = 40 $\mu$m

330 $\mu$m

100 $\mu$m

100 $\mu$m
Application III: Micro-crack initiation in a ceramic specimen (first attempt)

Evolution of damage $D$ (before the peak load)

$D_{max}$ is attained on the grain boundary

Micro-crack nucleates inside of the specimen, namely, on the grain boundary
New Error Estimator: conclusions

\[ \|e\|_\Omega \leq \frac{\hat{C}_p \hat{C}_K}{\sqrt{2\mu + \hat{C}_{SE} \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}} \]

\[ \eta_{T,\text{New}} := h_T \|f + \text{div}\sigma(u^h)\|_{L^2(T)} + \frac{h_T}{T^{1/2}} \sum_{\ell=1}^3 \|E_{\ell}^{1/2}\|_{E_{\ell}} \|\sigma(u^h) \cdot n\|_{L^2(E_{\ell})} \]

- **Explicit** error estimator for 2D (linear) problems
- **Simple** and **cheap**
  - *UB* is **guaranteed** (no overestimation)
  - *UB* is **accurate** (eff.ind. < 2 => practically acceptable)

- Extension to 3D (linear) problems is seemingly straightforward

New Error Estimator: ongoing research

\[
\|e\|_\Omega \leq \frac{\bar{c}_p \bar{c}_K}{\sqrt{2\mu + \bar{c}_{SE}^\lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}}
\]

\[\eta_{T,\text{New}} := h_T \| f + \text{div}\sigma(u^h) \|_{L^2(T)} + \frac{h_T}{|T|^{1/2}} \sum_{\ell=1}^3 \| E_\ell \|_{1/2}^{1/2} \| \sigma(u^h) \cdot n \|_{E_\ell} \|_{L^2(E_\ell)} \]

- Extension to \( P2, \ldots \) and \( Q1, Q2, \ldots \) based FEMs in 2D
- Extension to 3D
- Goal-oriented EE analysis, \( |Q(u) - Q(u^h)| \leq \| e \|_\Omega \| e^* \|_\Omega \)
- Extension to e.g. XFEM – done in [1], but constant-free (???)

Applications: conclusions

I. II. III.

A simple (lower-order) FE technique, equipped with an adequate a posteriori error estimator (to provide efficient, simple and cheap adaptivity), may be rather competitive to the advanced, yet more cumbersome and not-so-easy-to-implement FE techniques like e.g. XFEM/GFEM
Appendices

(planed for the next seminar)

• How does $C$ appears?

$$\|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}}$$
Appendices
(planed for the next seminar)

• How does $C$ appears?

\[ \|e\|_\Omega \leq \frac{C}{\sqrt{2\mu}} \left( \sum_T \eta_{T,\text{Class}}^2 \right)^{1/2} =: UB_{\text{Class}} \]

• Derivation of

\[ \|e\|_\Omega \leq \frac{\tilde{c}_p \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_SE \lambda}} \left( \sum_T \eta_{T,\text{New}}^2 \right)^{1/2} =: UB_{\text{New}} \]

\[ \eta_{T,\text{New}} := h_T \|f + \text{div}\sigma(u^h)\|_{L^2(T)} + \frac{h_T}{T^{1/2}} \sum_{\ell=1}^3 \|s\|_{E_\ell} \|f^h \cdot n\|_{E_\ell} \]

\[
\tilde{c}_p = \frac{4(\sqrt{17} - 1)^{1/2}}{(7 + \sqrt{17})(3 + \sqrt{17})^{1/2}} \\
\tilde{c}_K = 2\left( \frac{\pi}{3\pi + 2} \right)^{1/2} \\
\tilde{c}_{SE} = \frac{2\pi + 4}{3\pi + 2}
\]