Institut für
Kontinuummechanik

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Universität
Hannover

New simple, cheap and efficient error estimator for adaptive fracture and damage analysis (in 2D)

Tymofiy
Gerasimov

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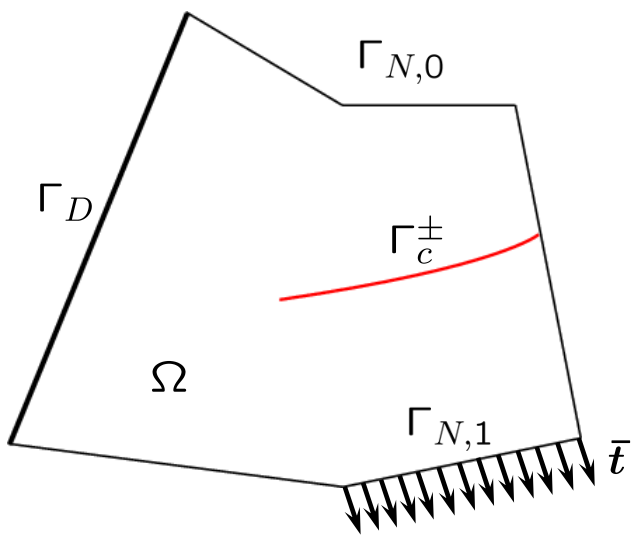
New simple, cheap and efficient error estimator for adaptive fracture and damage analysis (in 2D)

- Background: what is error estimator (EE)?
- Example of EE, how it works
- Motivation/Main result
- Applications: fracture & damage analysis

Background

What is error estimator?



Problem strong formulation:

$$
\begin{cases}
-\text{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} & \text{in} \quad \Omega \\
\boldsymbol{\sigma} = \mathcal{C} : \varepsilon(\boldsymbol{u}) & \text{in} \quad \Omega \\
\varepsilon = \boldsymbol{\nabla}^{\text{sym}}\boldsymbol{u} & \text{in} \quad \Omega \\
\boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \Gamma_D \\
\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma_{N,0} \\
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\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}^{\pm} = \boldsymbol{0} & \text{on} \quad \Gamma_c^{\pm}
\end{cases}
$$

Background

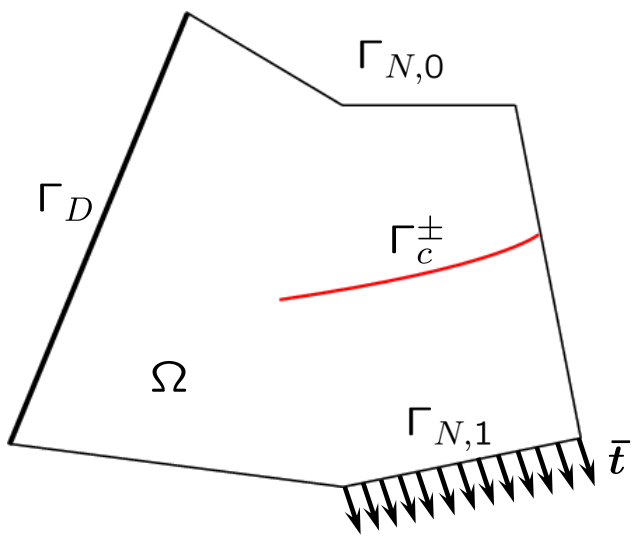
What is error estimator?

discr. error

$$\mathbf{u} - \mathbf{u}_h = \mathbf{e}$$

exact solution (unknown)

FE solution (available)



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$$\left\| \mathbf{u} - \mathbf{u}_h \right\|_\Omega = \left\| \mathbf{e} \right\|_\Omega$$

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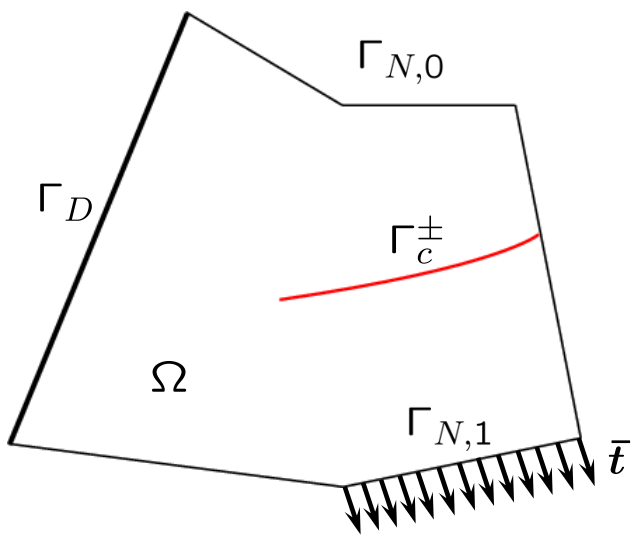

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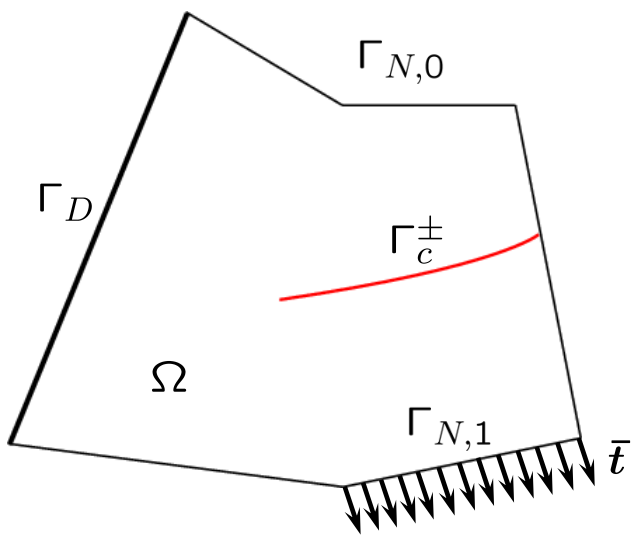
Background

What is error estimator?

discr. error

$$\left\| \mathbf{u} - \mathbf{u}_h \right\|_\Omega = \left\| \mathbf{e} \right\|_\Omega \leq UB$$

I. To assess the **accuracy** of your FE solution by computing the *UB*



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Background

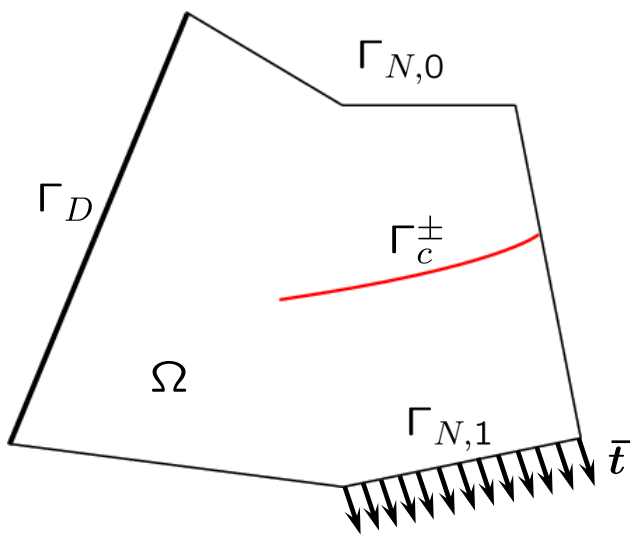
What is error estimator?

discr. error

$$\left\| \mathbf{u} - \mathbf{u}_h \right\|_\Omega = \left\| \mathbf{e} \right\|_\Omega \leq UB$$

I. To assess the **accuracy** of your FE solution by computing the *UB*

II. To trigger a refinement strategy (e.g. **mesh refinements**) => better **accuracy**



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Background

What is error estimator?

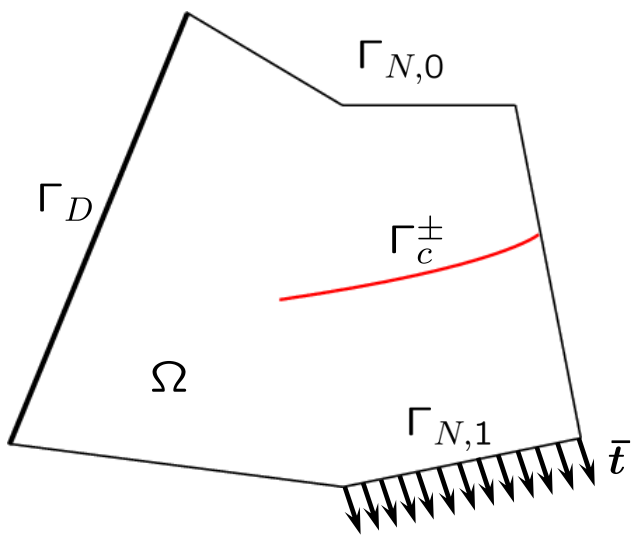
local error indicators

discr. error

$$\left\|\mathbf{u}-\mathbf{u}_h\right\|_\Omega = \left\|\mathbf{e}\right\|_\Omega \le UB = \left(\sum_T \eta_T^2\right)^{1/2}$$

I. To assess the **accuracy** of your FE solution by computing the *UB*

II. To trigger a refinement strategy (e.g. **mesh refinements**) => better **accuracy**



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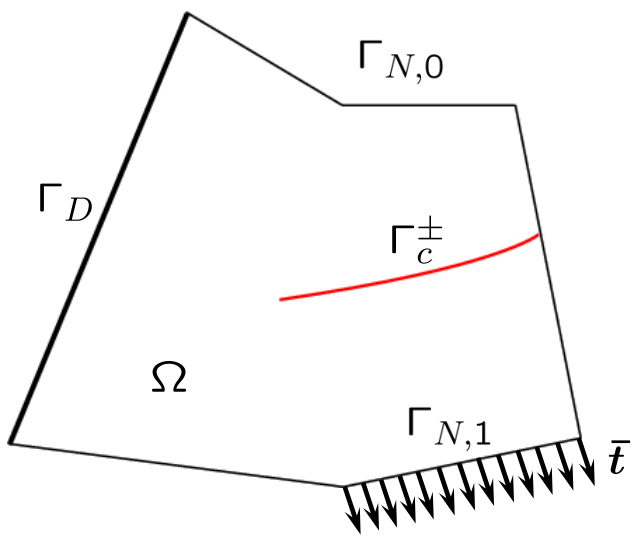
Background

What is error estimator?

$$\boxed{\left\|\!\left|\mathbf{e}\right|\!\right\|_{\Omega} \le UB = \left(\sum_{T} \eta_T^2\right)^{1/2}}$$

Computable Upper Bound

local error indicators



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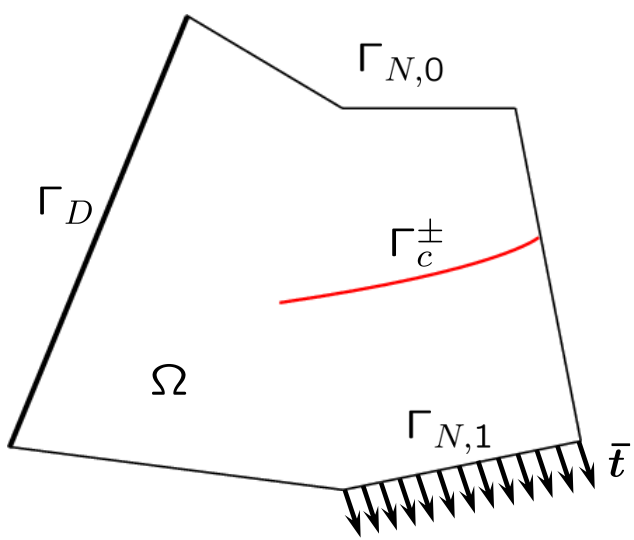
Example

Available classical Babuška-Miller estimator:

$$\left\|\mathbf{e}\right\|_{\Omega} \leq \frac{C}{\sqrt{2\mu}}\left(\sum_{T}\eta_{T,Class}^{2}\right)^{1/2} =: UB_{Class}$$

$$\eta_{T,Class} := h_T\left\|\mathbf{f}+\mathrm{div}\boldsymbol{\sigma}(\mathbf{u}^h)\right\|_{\mathbf{L}^2(T)} + h_T^{1/2}\sum_{\ell=1}^{3}\left\|\left\langle\boldsymbol{\sigma}(\mathbf{u}^h)\cdot\mathbf{n}\right\rangle_{E_\ell}\right\|_{\mathbf{L}^2(E_\ell)}$$



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Derived straightforwardly by using the theory [1], [2], [3]

[1] I. Babuška, W.C. Rheinboldt "A posteriori error estimates for the finite element method", *IJNME* 1978; **12**:1597–1615.

[2] D.W. Kelly et al. "A posteriori error analysis and adaptive processes in the finite element methods: Part I – Error analysis", *IJNME* 1983; **19**:1953–1619.

[3] I. Babuška, A. Miller "A feedback finite element method with a posteriori error estimation: Part I. The finite element method and some basic properties of the a posteriori error estimator", *CMAME* 1987; **61**:1–40.

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diam(T)

Example

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diam(T)

a) **Explicit** (all data involved is explicitly available) => **simple** and **cheap**

Example

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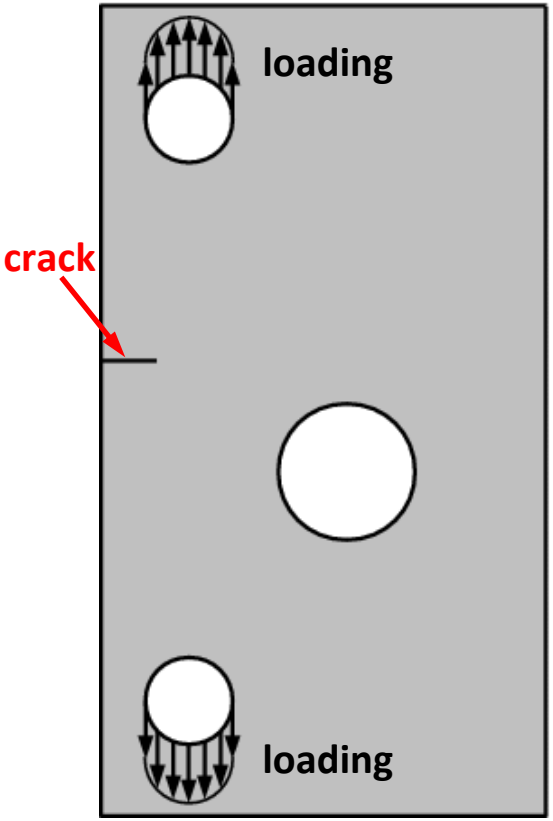
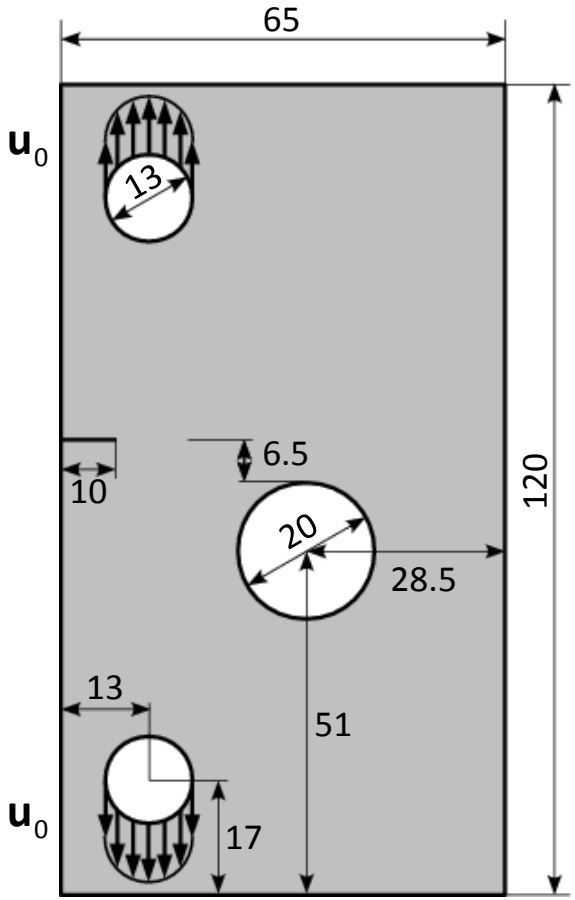
diam(T)

a) **Explicit** (all data involved is explicitly available) => **simple** and **cheap**

b) Basis for **adaptive** mesh refinements => you refine, where it is **appropriate**

Example





(all dimensions in *mm)*

plane strain

$$E = 77.1 \ GPa$$
$$\nu = 0.33$$

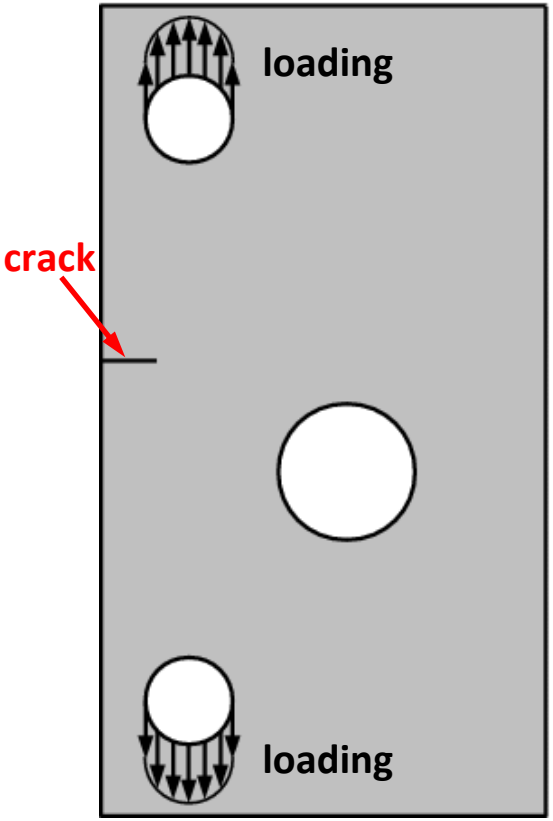
(aluminum 7075-T6)

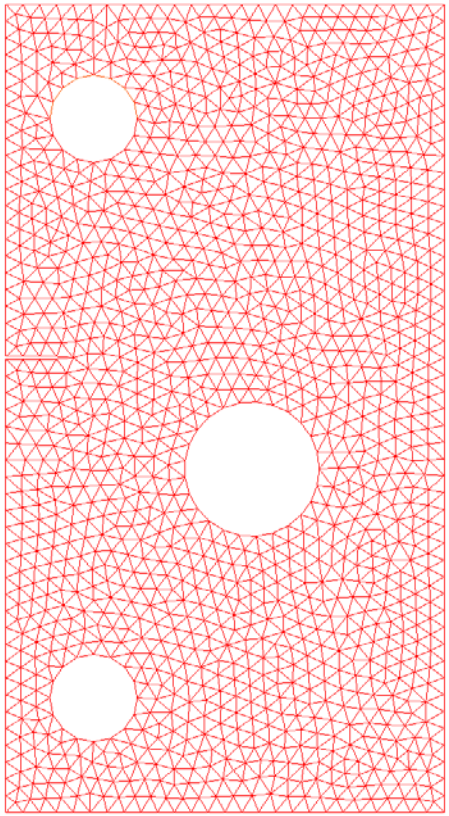
displacement controlled loading with

$$\mathbf{u}_0 = (0, 1.56 \cdot 10^{-5}) \ mm$$

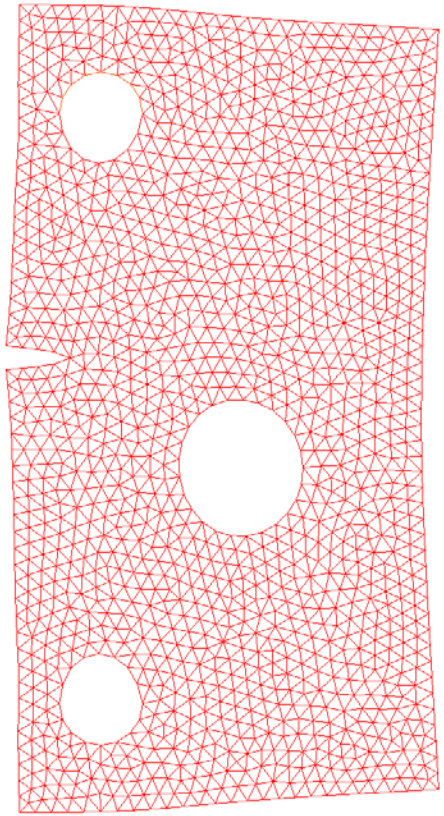
*P*1-triangular FEM

Example





Initial coarse mesh
(2854 DOF)



Deformed mesh
(magn. $1.5 \cdot 10^5$)

Example

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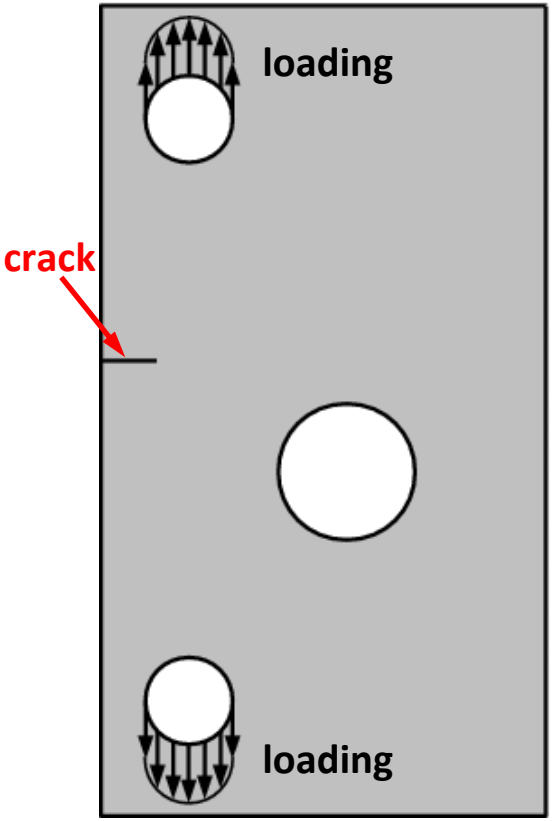

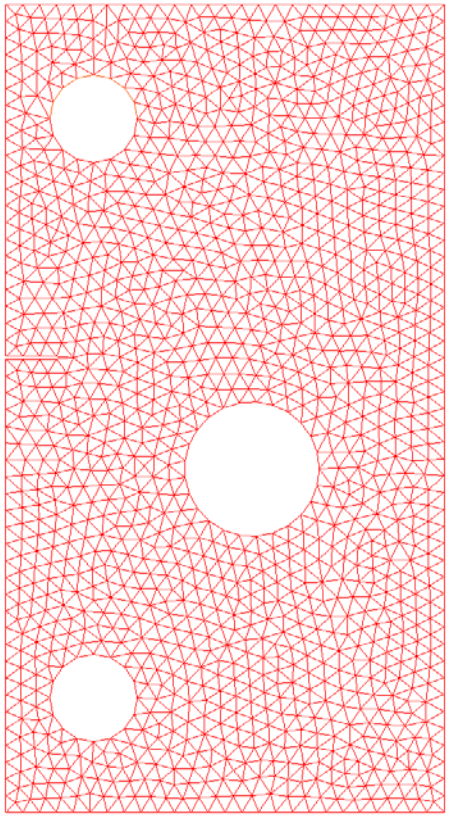

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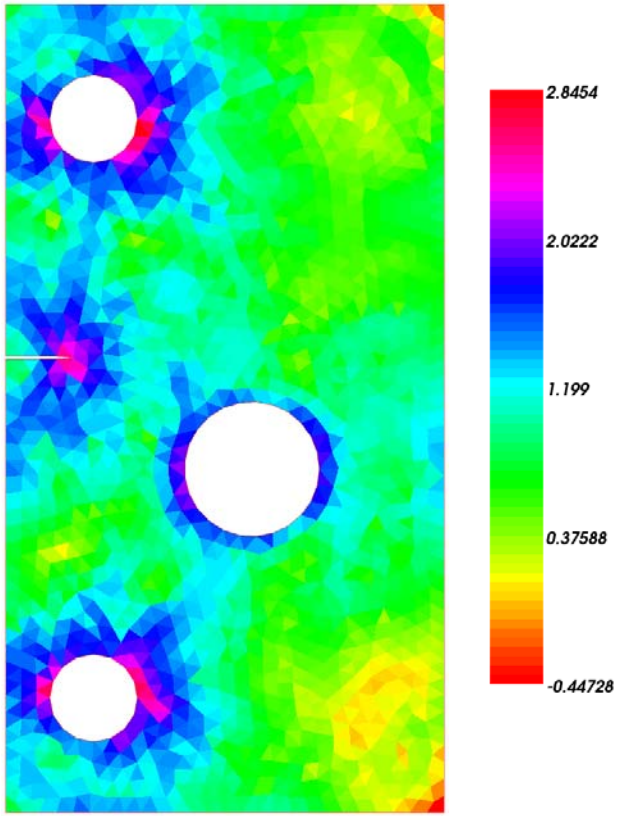
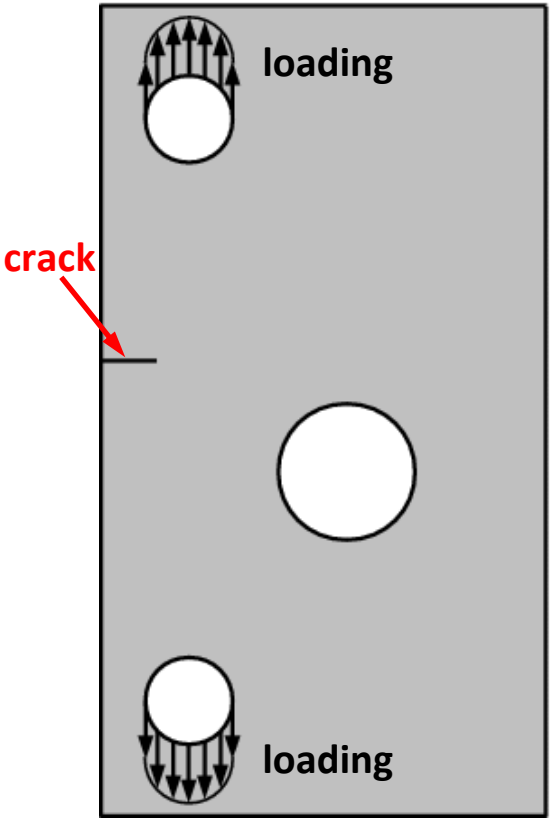
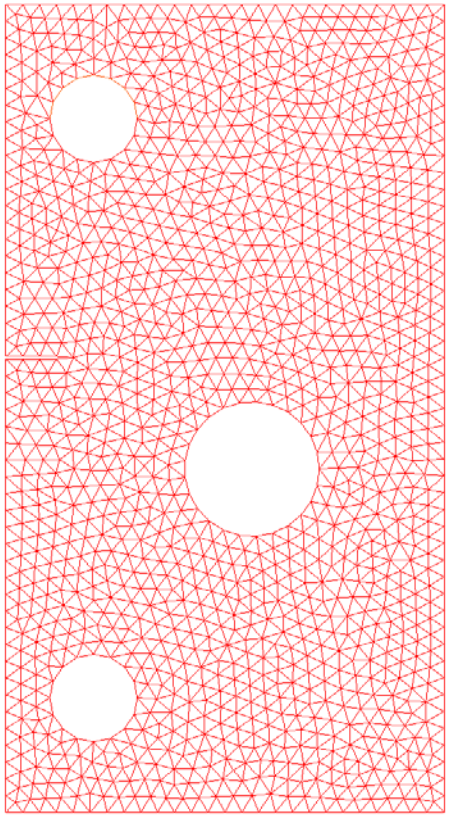


Distribution of
local errors,

$\log_{10}(\eta_{T,Class})$

Example





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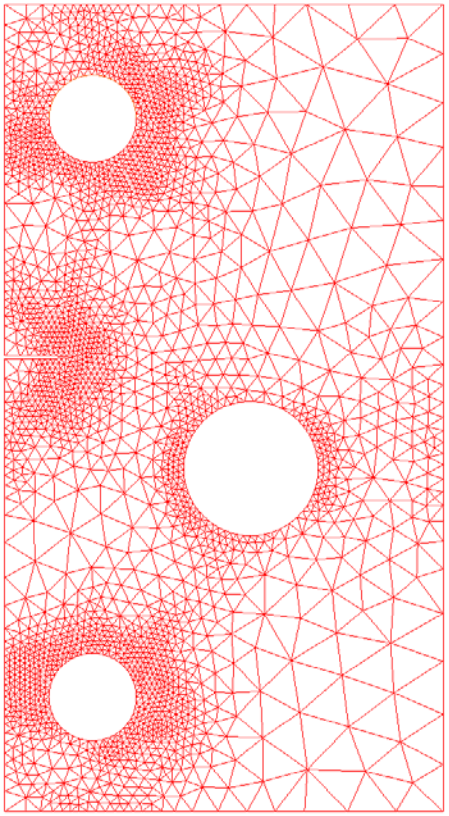
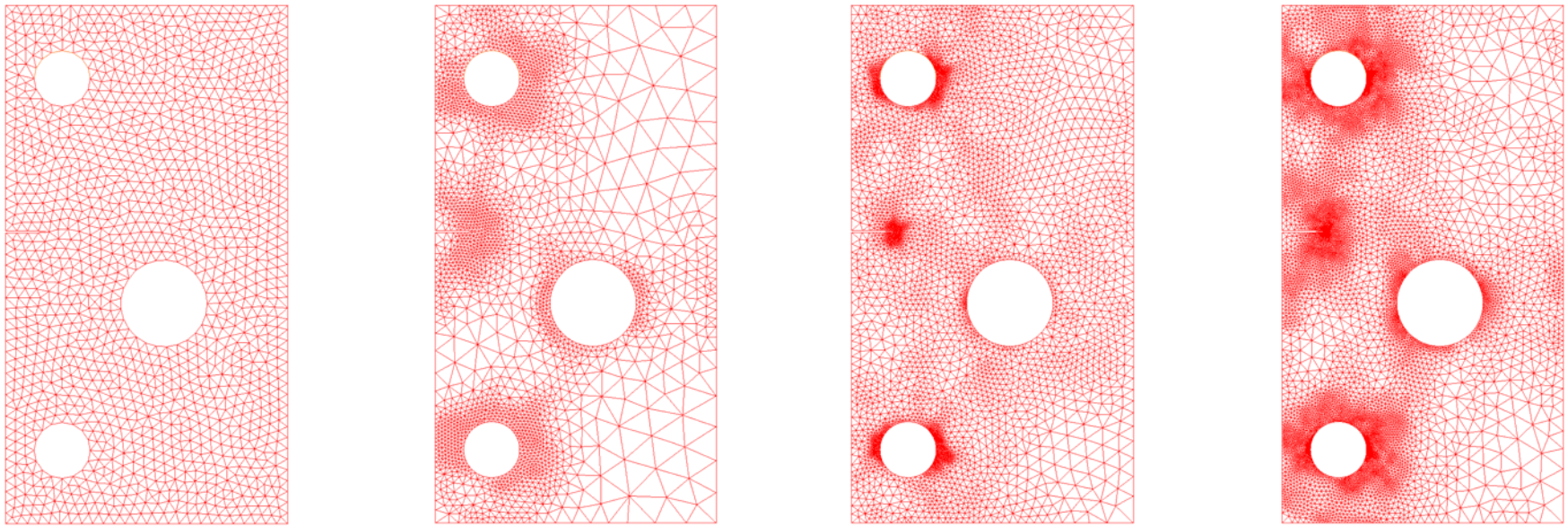
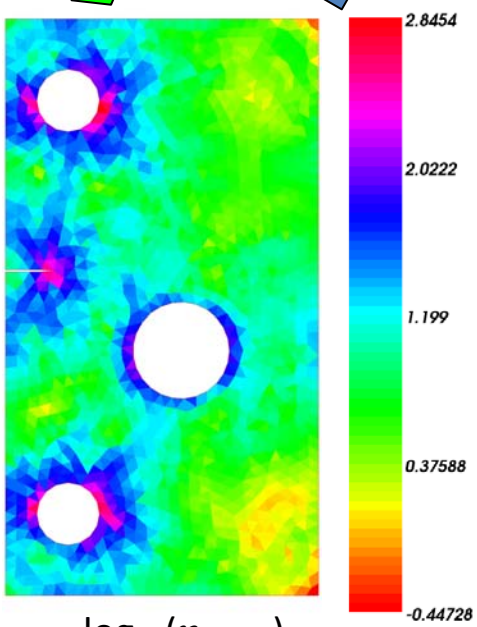


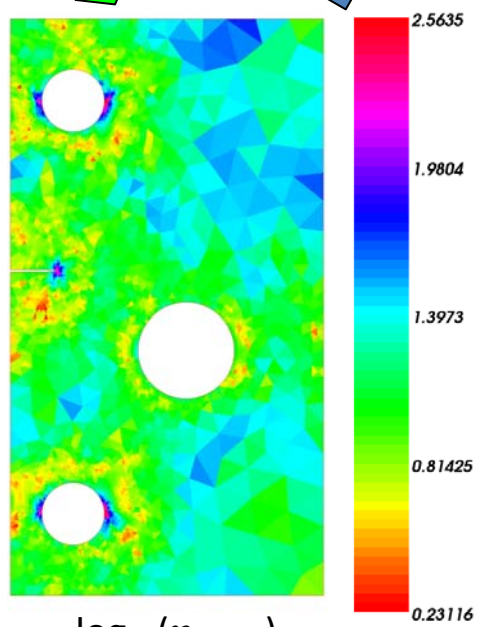
Adaptive step 1
(5078 DOF)

Example





$\log_{10}(\eta_{T,Class})$



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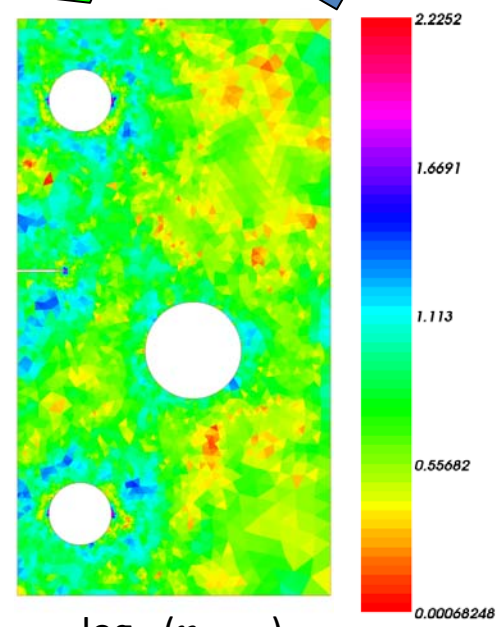

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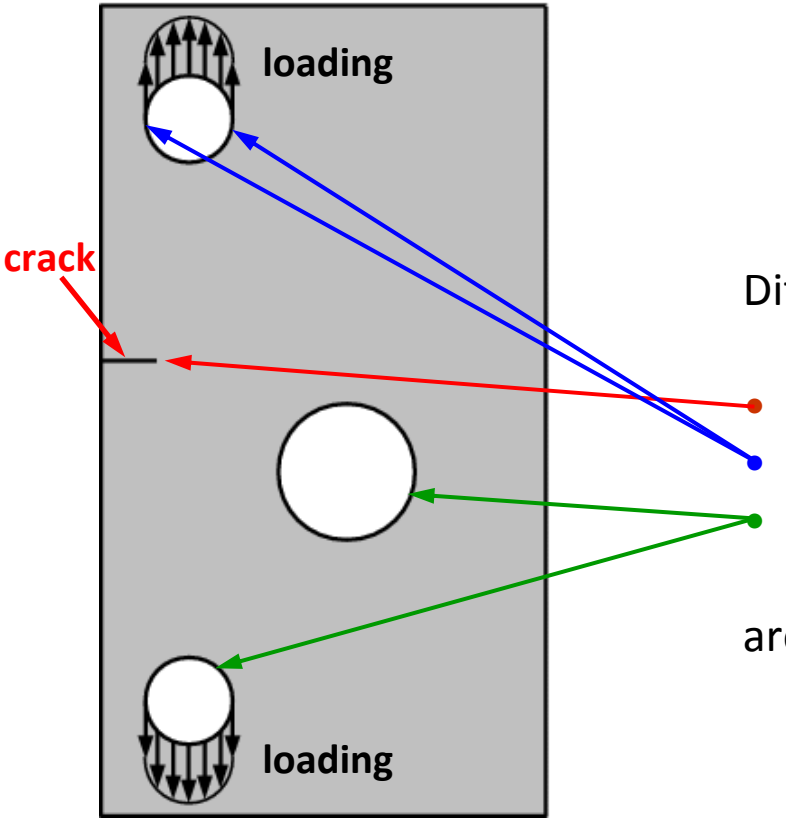
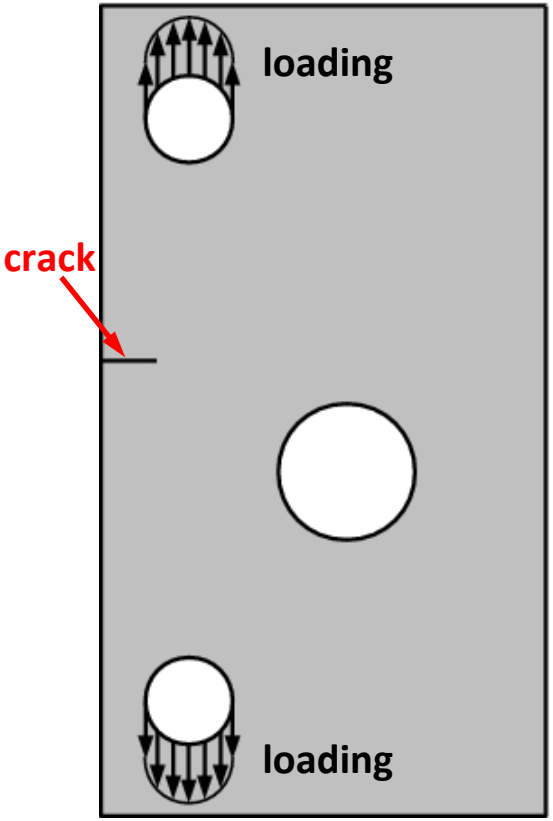


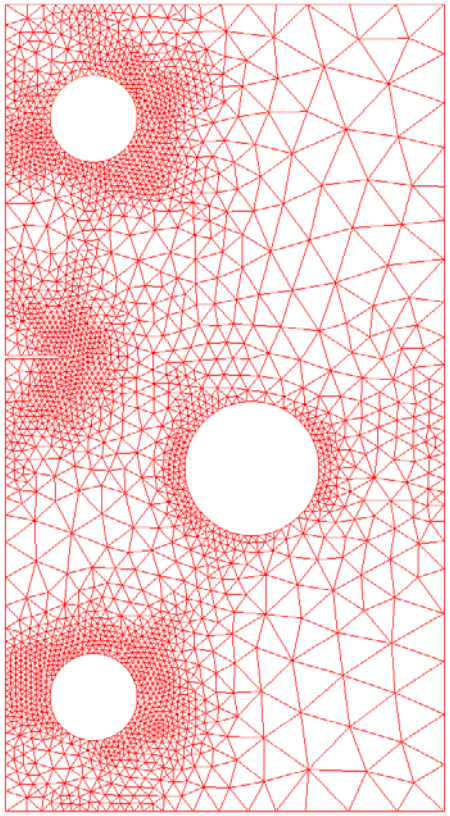
Different types of (strong/weak) singularities:

- Crack tip (limiting case of a re-entrant corner)
- The points, where the BC change
- Concave parts of the boundary

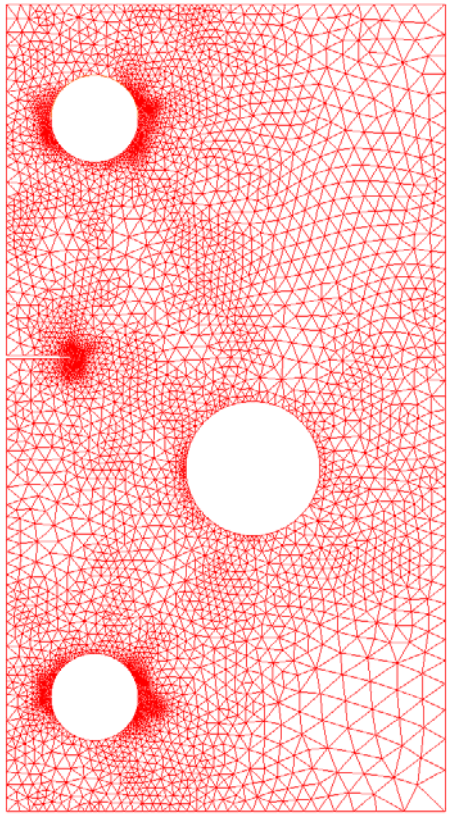
are naturally captured by mesh adaptivity

Example





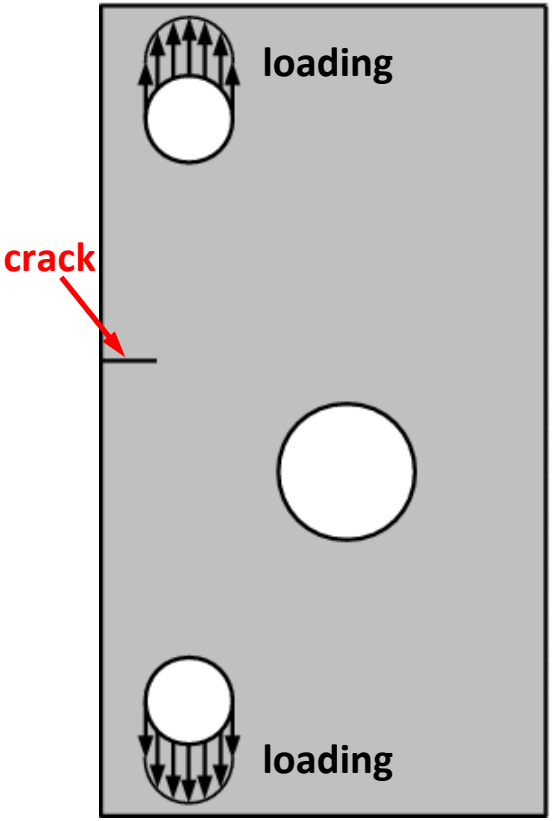
Adaptive step 1
(5078 DOF)

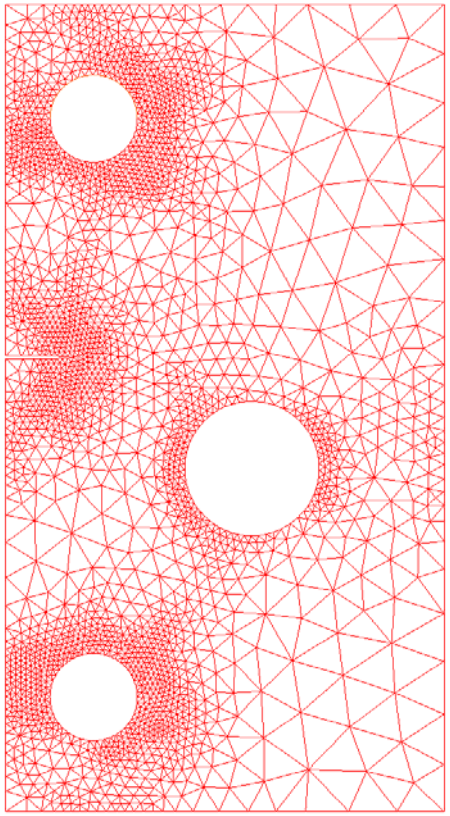


Adaptive step 2
(9682 DOF)

etc.

Example





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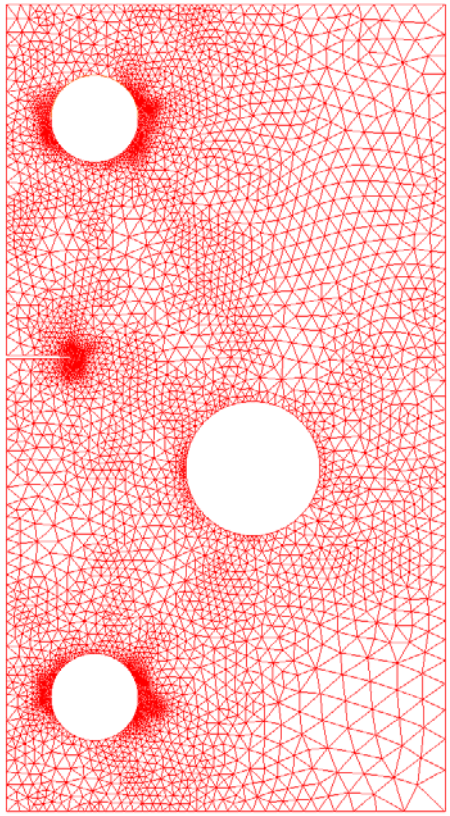

Adaptive step 2
(9682 DOF)

?

etc.

(until **when** is to refine ?)

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$$\eta_{T,Class} := h_T\left\|\mathbf{f}+\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}^h)\right\|_{\mathbf{L}^2(T)} + h_T^{1/2}\sum_{\ell=1}^{3}\left\|\left\langle\boldsymbol{\sigma}(\mathbf{u}^h)\cdot\mathbf{n}\right\rangle_{E_\ell}\right\|_{\mathbf{L}^2(E_\ell)}$$

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a) **Explicit**, **simple** and **cheap**
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Multiplicative constant C is **not known**

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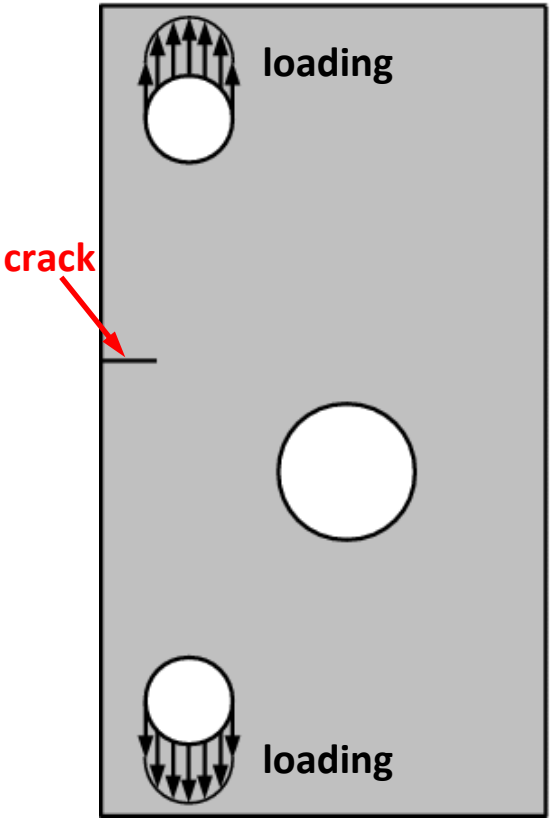
By no means C can be found through the **actual** derivation procedure

Example

What if to set C=1 ?



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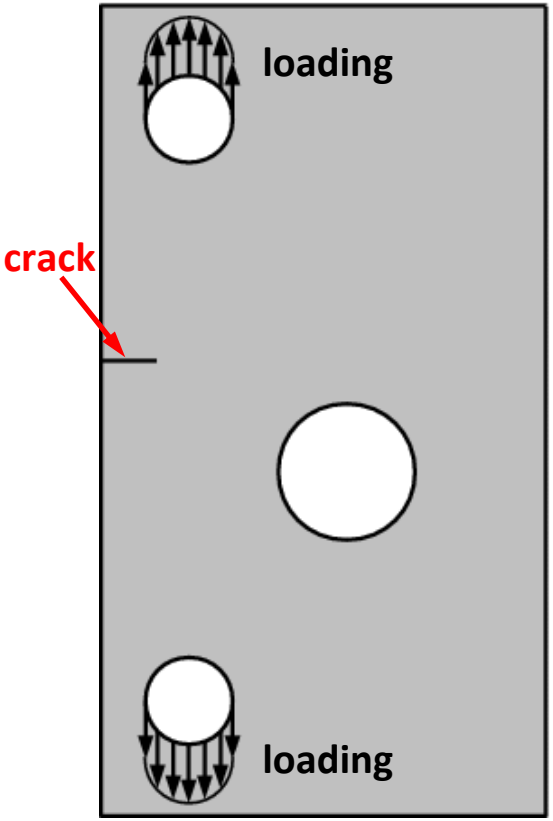

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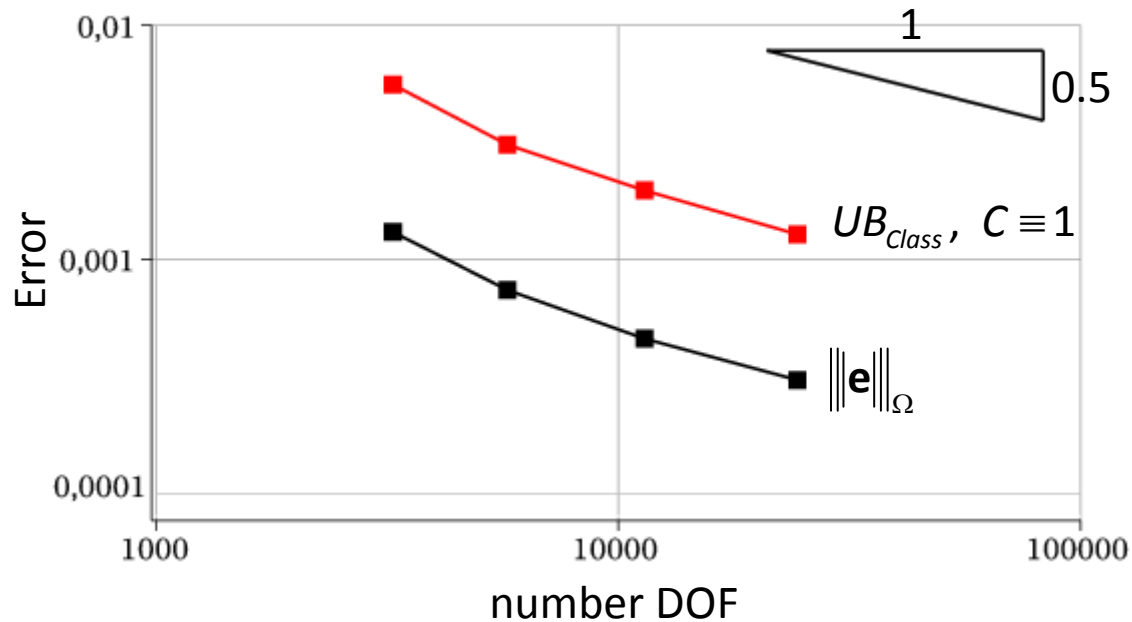

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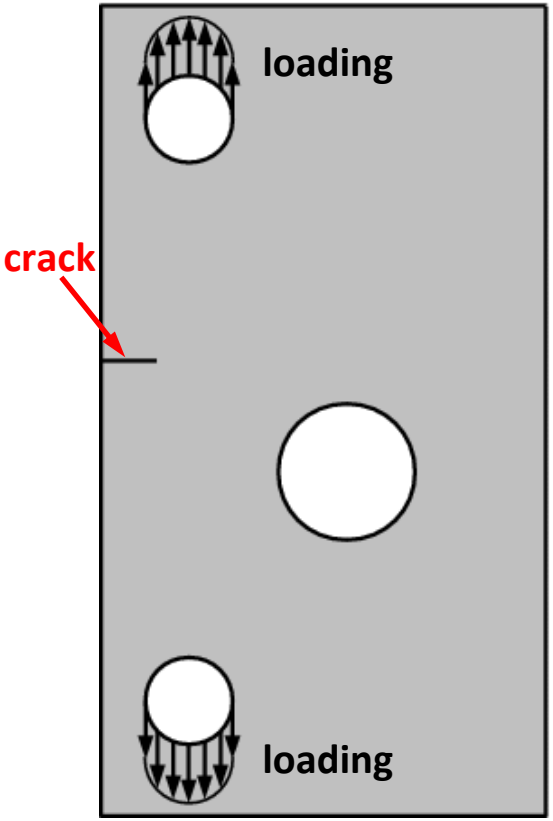

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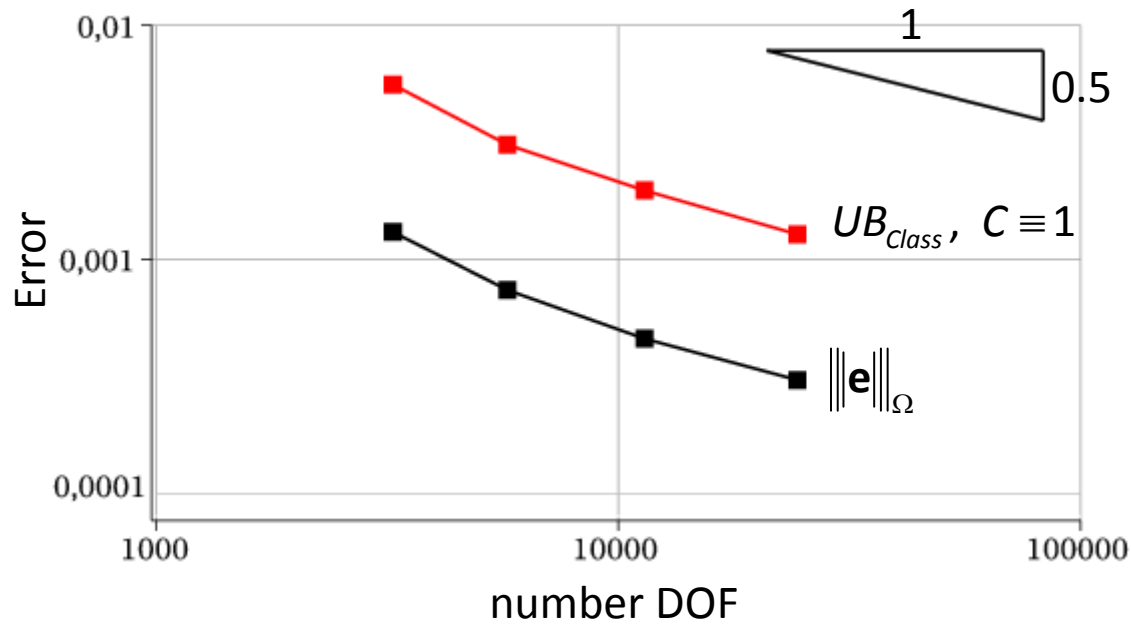

NB: reference ("exact") error $\left\|\!\left|\mathbf{e}\right|\!\right\|_{\Omega}$ obtained through computing an overkill solution

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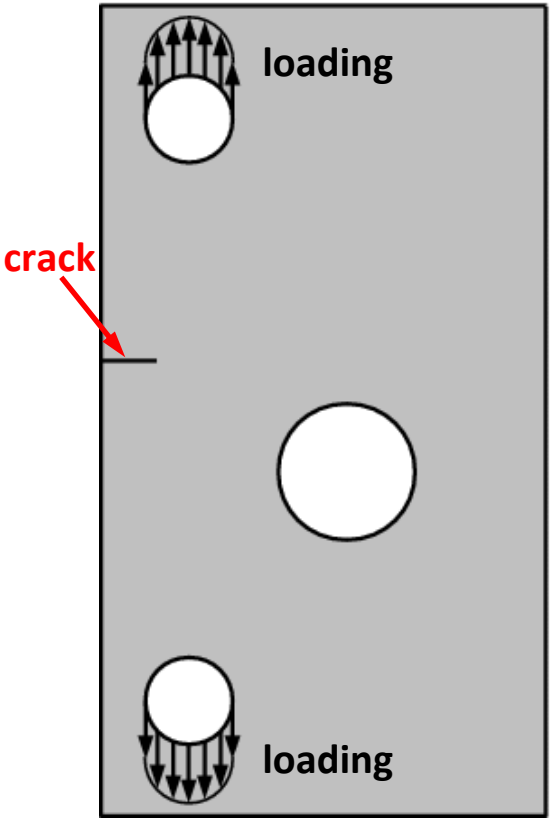

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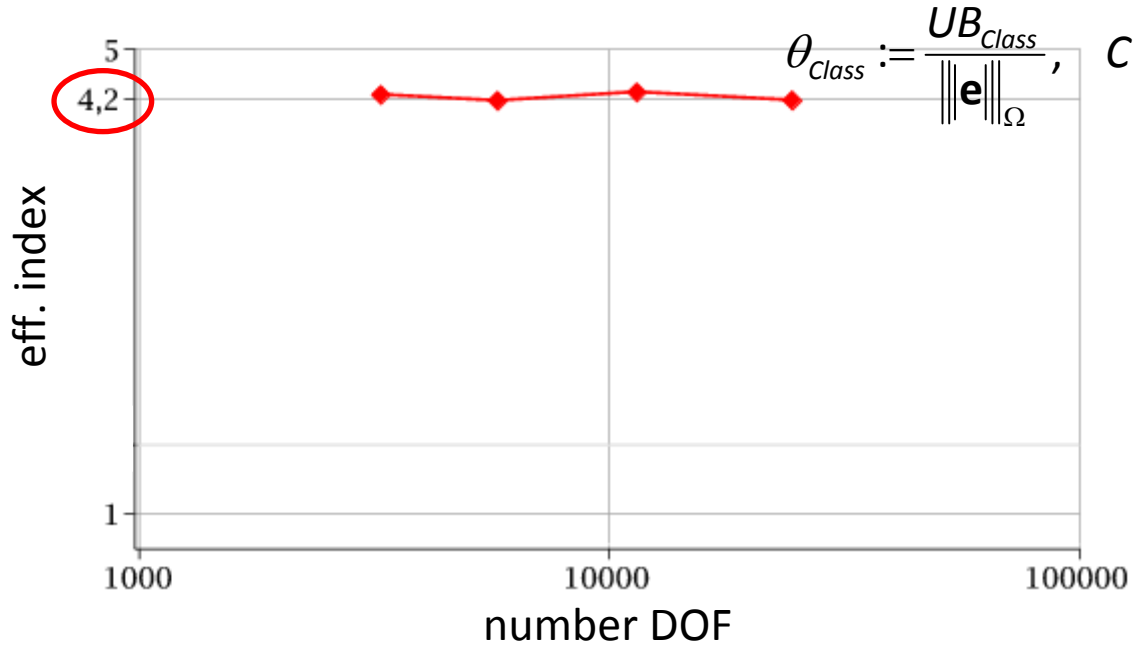

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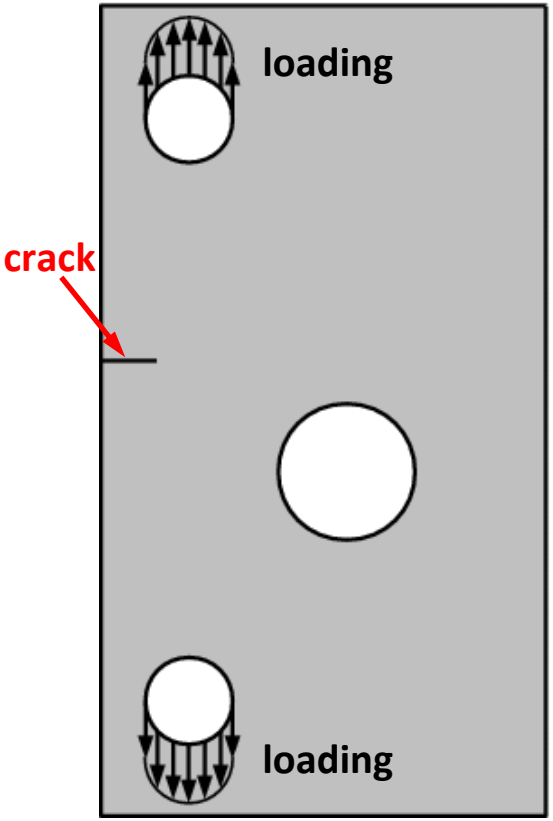

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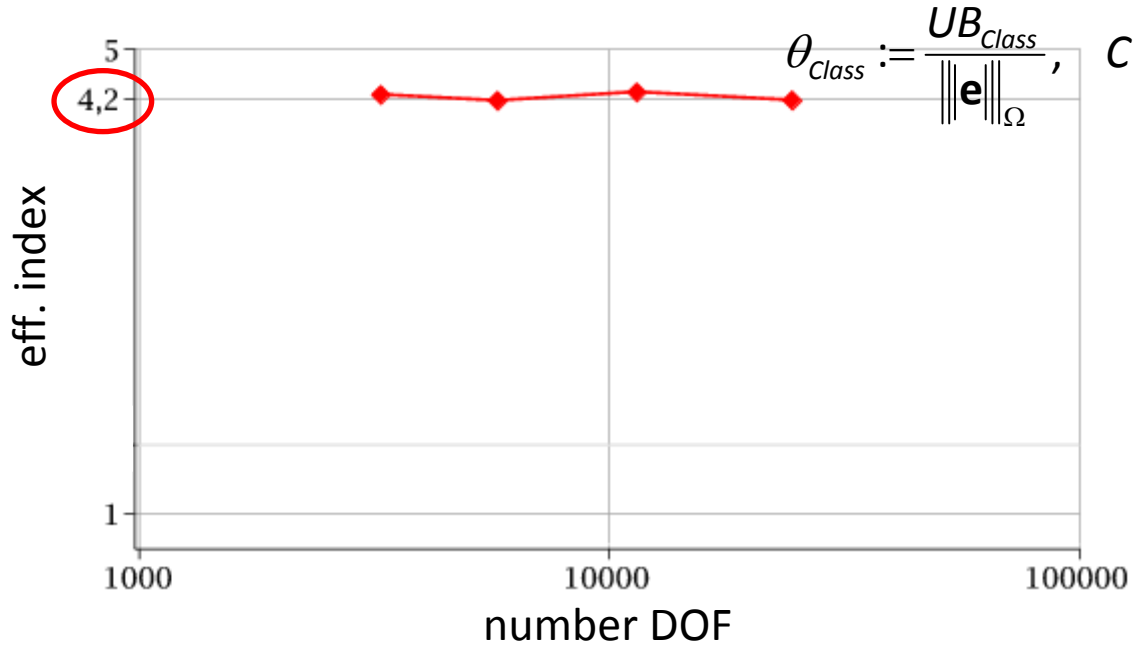

Is not allowed; if done – practically **useless** estimator

Motivation / Main result

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My motivation: Pros ! ~~Cons~~

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$$\left|\!\left|\!\left|\mathbf{e}\right|\!\right|\!\right|_{\Omega} \leq \frac{\tilde{c}_P\tilde{c}_K}{\sqrt{2\mu+\tilde{c}_{SE}\lambda}}\left(\sum_{T}\eta_{T,New}^{2}\right)^{1/2} =: UB_{New}$$

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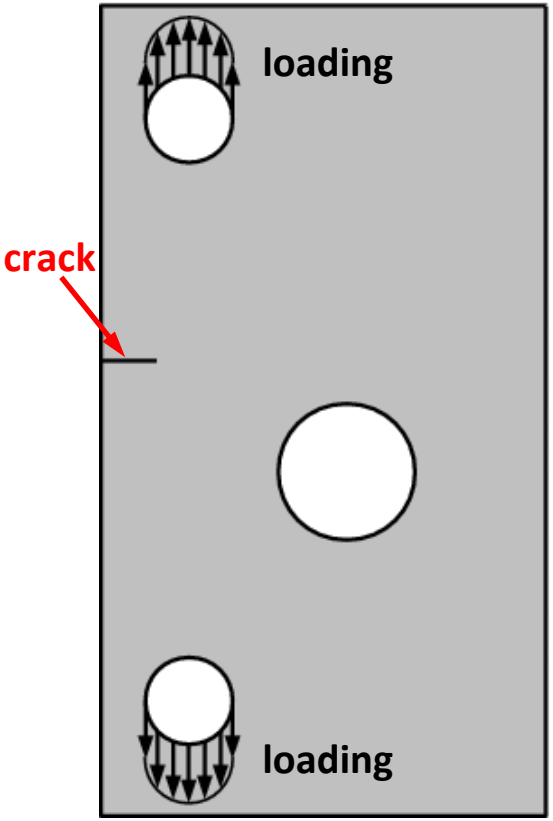

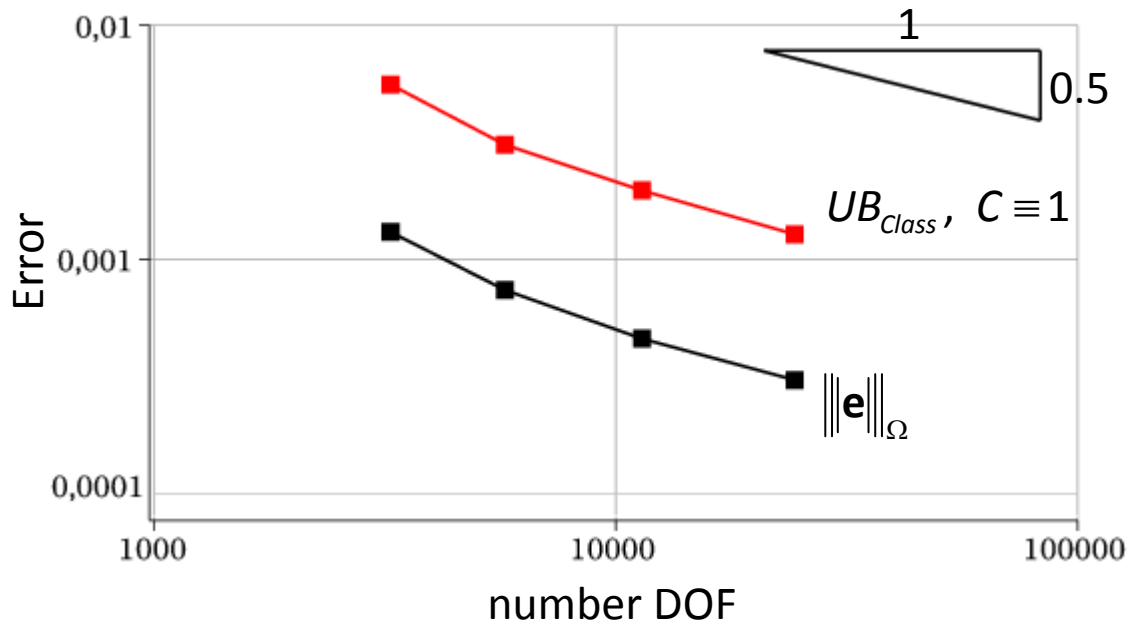

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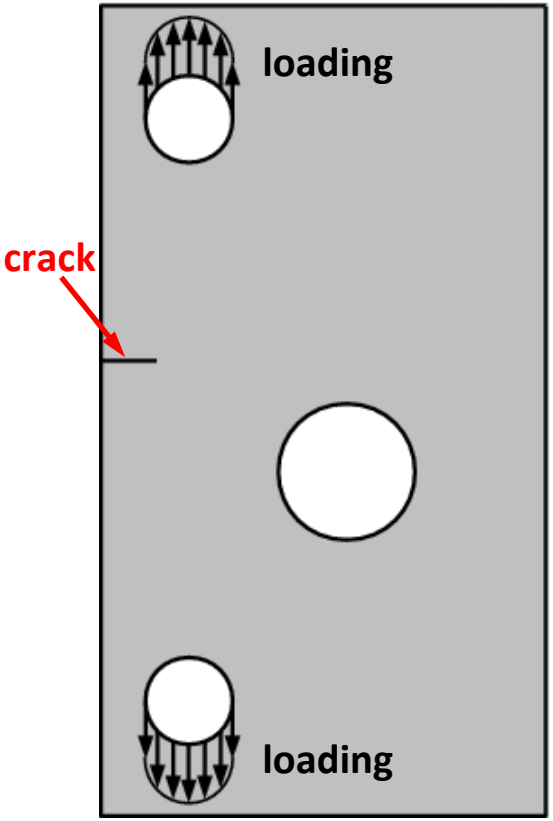

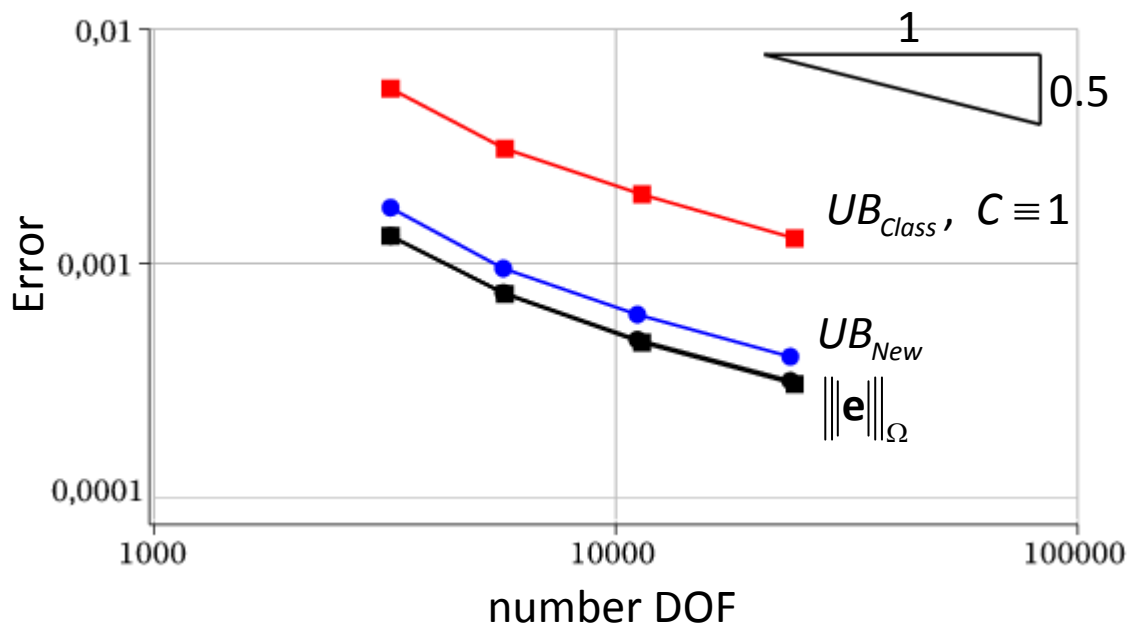

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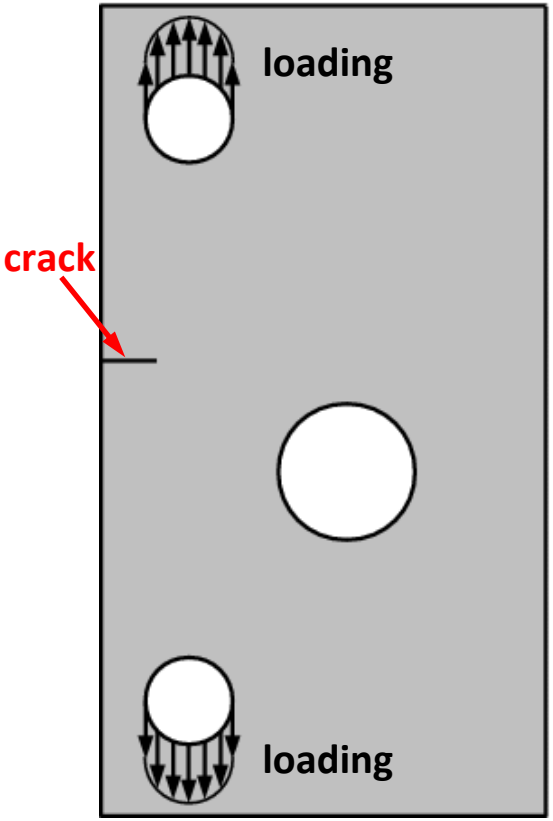

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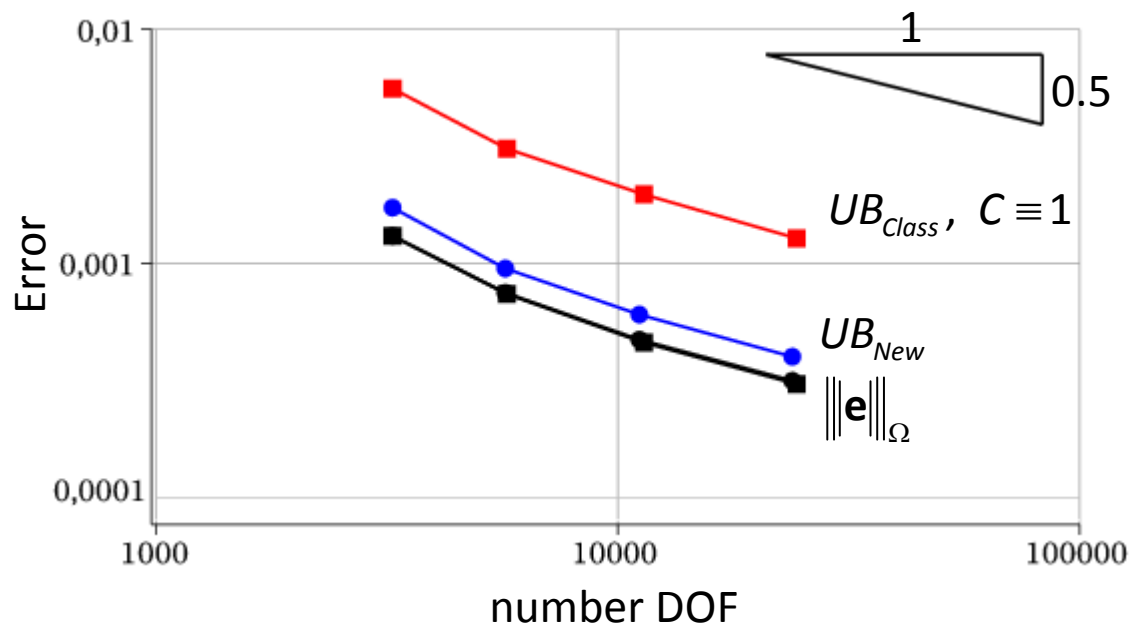
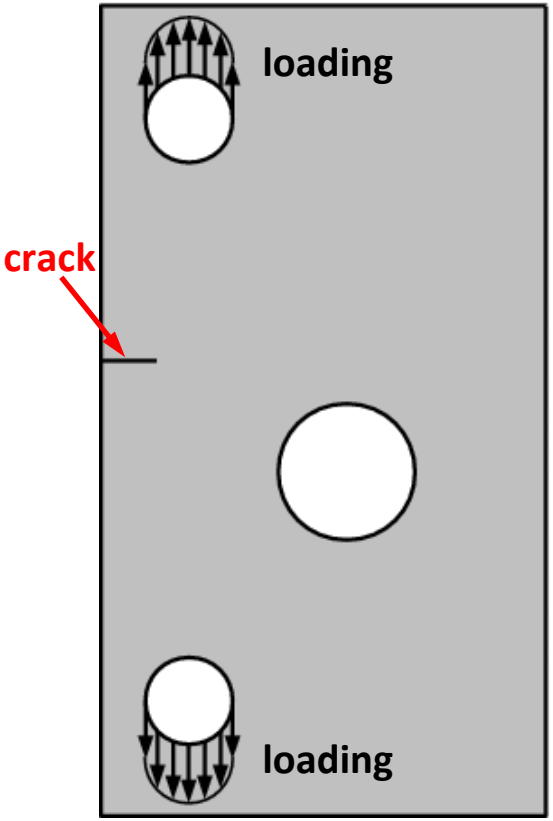


New Upper Bound is guaranteed and accurate

Motivation / Main result



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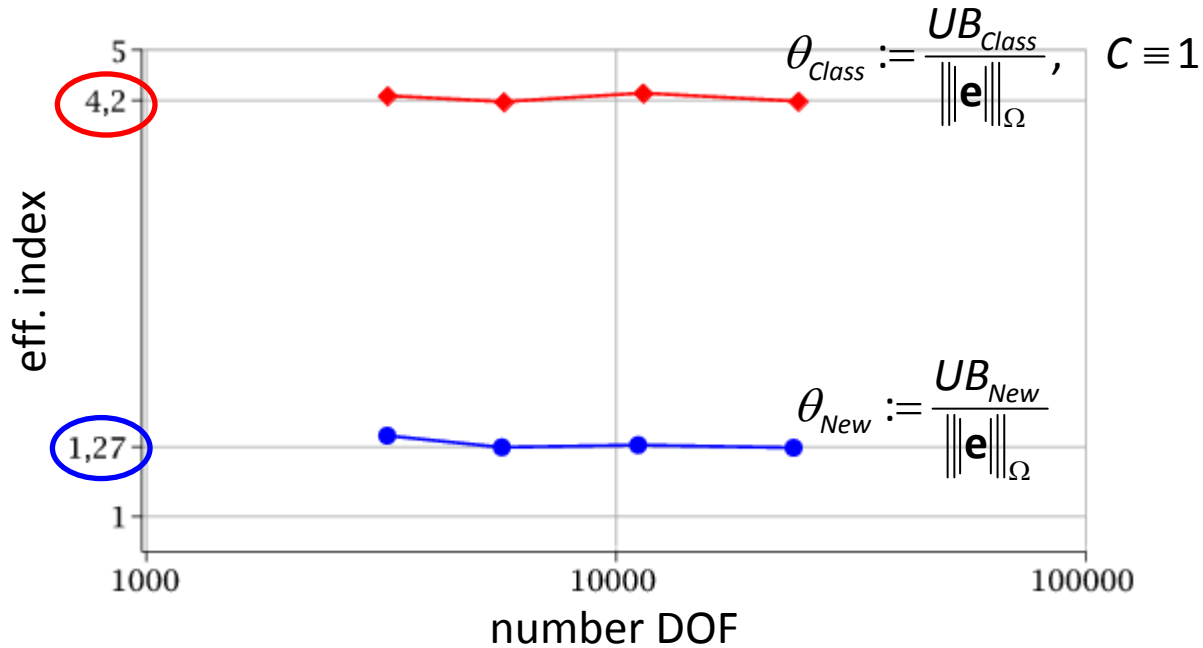
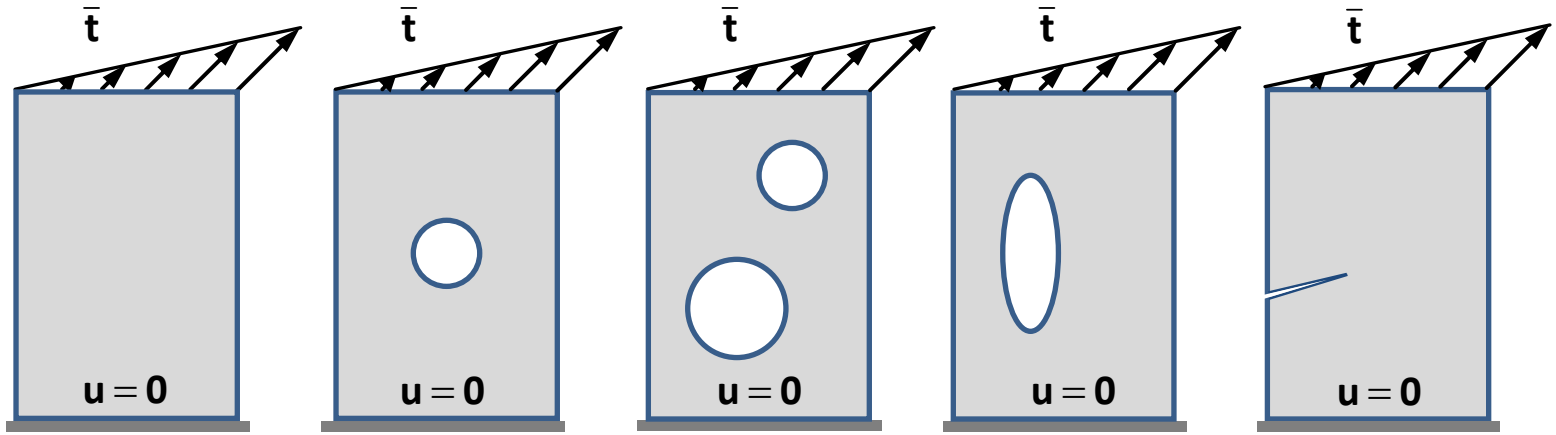


Corresponding **effectivity index** is close to 1 !

Motivation / Main result



plane stress

$E = 65000$

$\nu = 0.29$

The above geometries were used for the primal testing of

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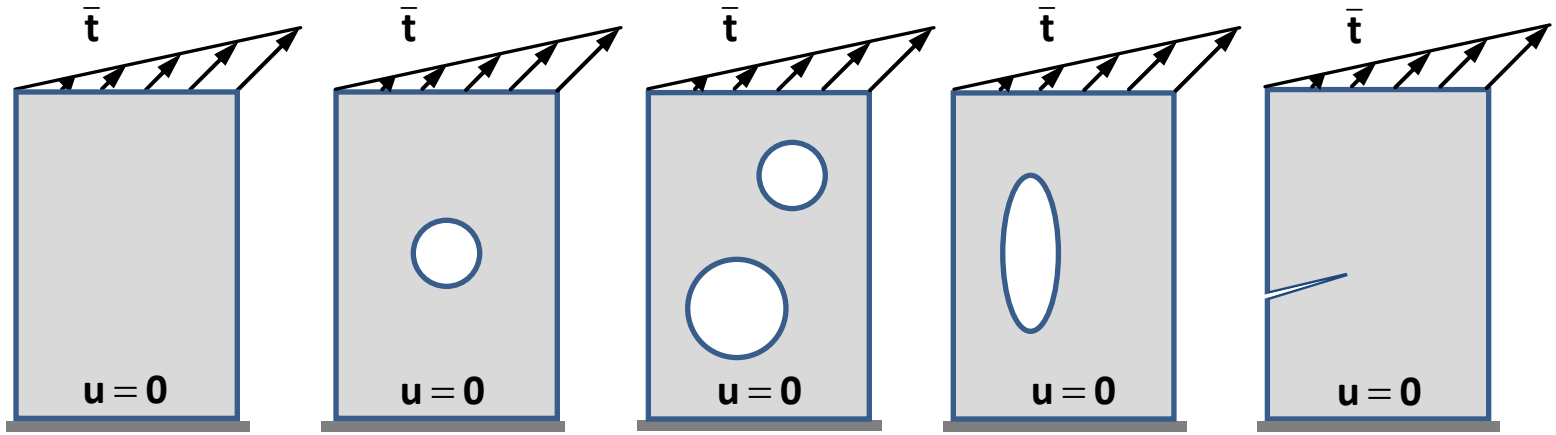
*P*1-triangular FEM

Motivation / Main result



1. In all examples considered $\theta_{New} := \dfrac{UB_{New}}{\|\mathbf{e}\|_{\Omega}} < 2$ (namely, 1.3-1.6)

   both for the **uniform** and **adaptive** mesh refinement strategies

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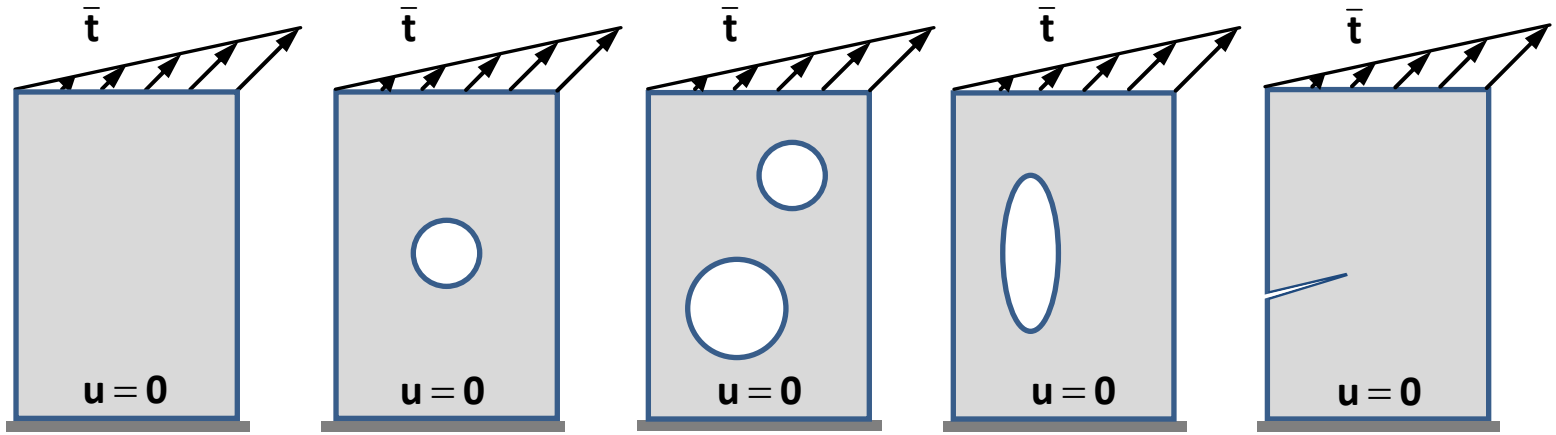

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   both for the **uniform** and **adaptive** mesh refinement strategies

*P*1-triangular FEM

2. In the letter case, for *p*=1 the **optimal** convergence rate (1 and 0.5 in terms of *h* and *N*, respectively) is **restored** (smt. **doubled**)

order of singularity

$$|||\mathbf{e}|||_{\Omega} \le C h^{\min(p,\lambda)} \qquad |||\mathbf{e}|||_{\Omega} \le \tilde{C} N^{\frac{1}{2}\min(p,\lambda)}$$

Motivation / Main result

Stopping criterion for mesh refinements (adaptive/uniform) is now “well-defined”:

$$\frac{\overset{?}{\left|\!\left|\!\left|\mathbf{e}\right|\!\right|\!\right|_{\Omega}^{2}}}{\overset{?}{a(\mathbf{u},\mathbf{u})}} \overset{!}{\leq} \mathrm{TOL}$$

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$$\left.\begin{aligned} \left|\!\left|\!\left|\mathbf{e}\right|\!\right|\!\right|_{\Omega} &\leq UB_{New} \\ a(\mathbf{u},\mathbf{u}) &= \left|\!\left|\!\left|\mathbf{e}\right|\!\right|\!\right|_{\Omega}^{2} + a(\mathbf{u}^{h},\mathbf{u}^{h}) \end{aligned}\right\}$$

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Application I:
Error-controlled crack propagation (in 2D)

Experimental setup [1]:

$E = 77.1 \ GPa$
$\nu = 0.33$
(aluminum 7075-T6)

[1] E. Giner et al. (2009) "An Abaqus implementation of the XFEM", *Engng. Fracture Mech.* **70**:347-368.

Application I:
Error-controlled crack propagation (in 2D)

Experimental setup [1]:

$E = 77.1 \ GPa$
$\nu = 0.33$
(aluminum 7075-T6)

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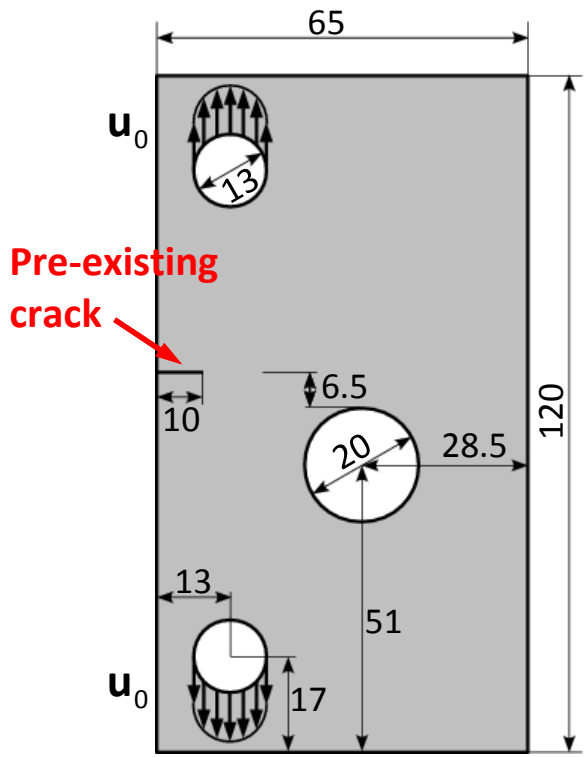

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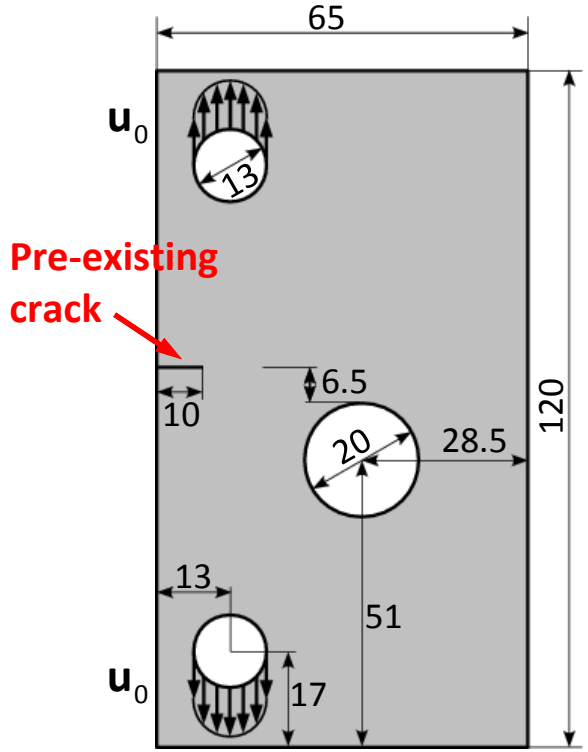

plane-strain

13 propagation steps, each with

- displ. $\mathbf{u}_0 = (0,\ 1.56 \cdot 10^{-5})\ mm$
- crack increment $\Delta a = 2.5\ mm$

*P*1-triangular FEM

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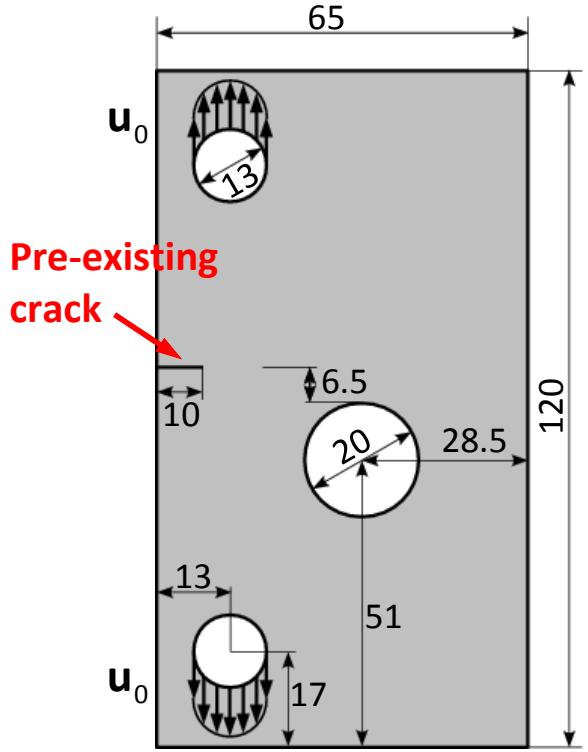

plane-strain

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*P*1-triangular FEM

- the domain expression for the *J*-integral:

$$J(\boldsymbol{u};\theta) = -\int_{\Omega_J} \boldsymbol{\nabla}(q\bar{\boldsymbol{n}}) : (W_s(\boldsymbol{u})\boldsymbol{I} - \boldsymbol{\nabla}^T\boldsymbol{u}\cdot\boldsymbol{\sigma}(\boldsymbol{u}))dxdy,$$

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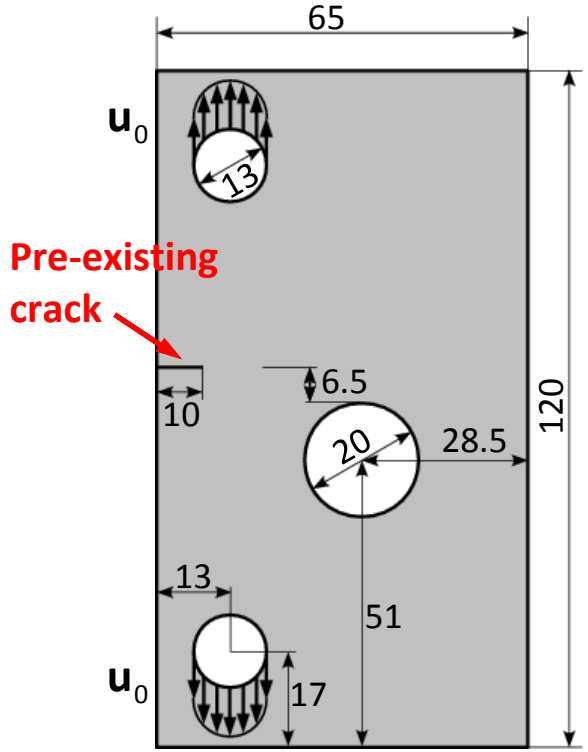
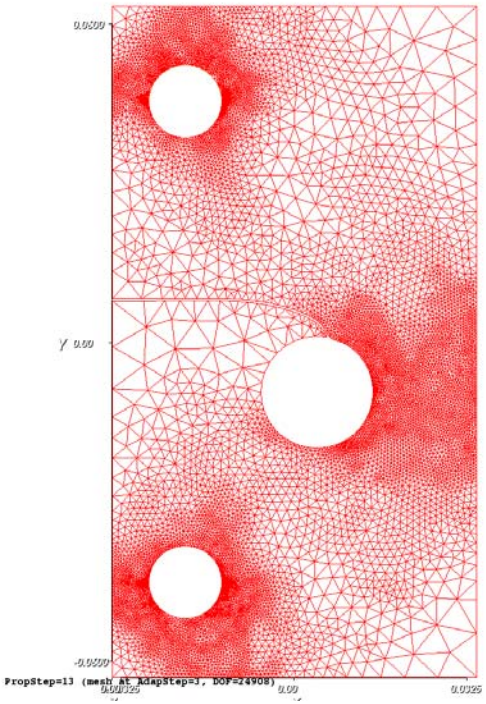


Our result [2]:

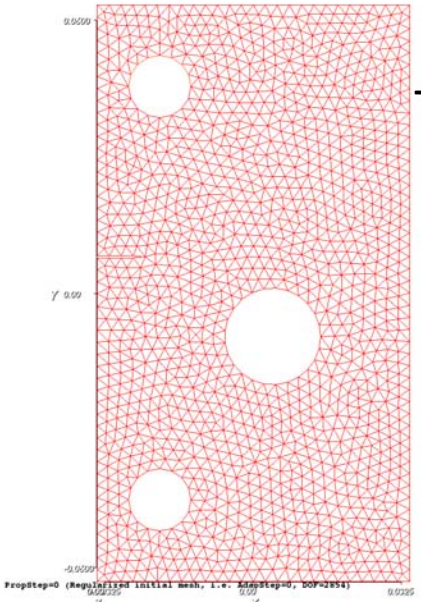


Final crack pattern

[2] T. Gerasimov, E. Stein and P. Wriggers (2013) Fully adaptive FEM/SFM for 2D crack propagation and damage analysis using a new simple constant-free explicit error estimator, *in preparation*.

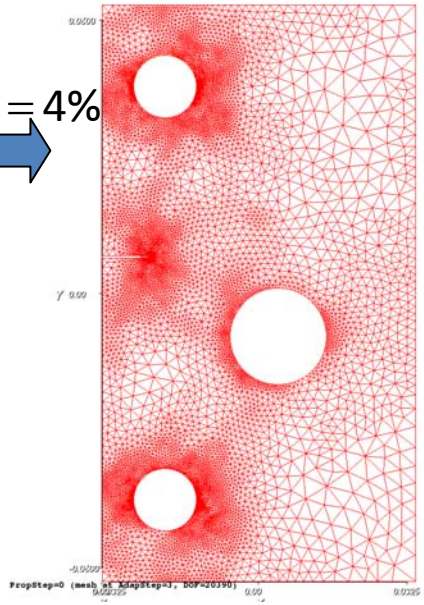
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 0
(pre-existing crack)



Initial mesh
(2854 DOF)

TOL = 4%



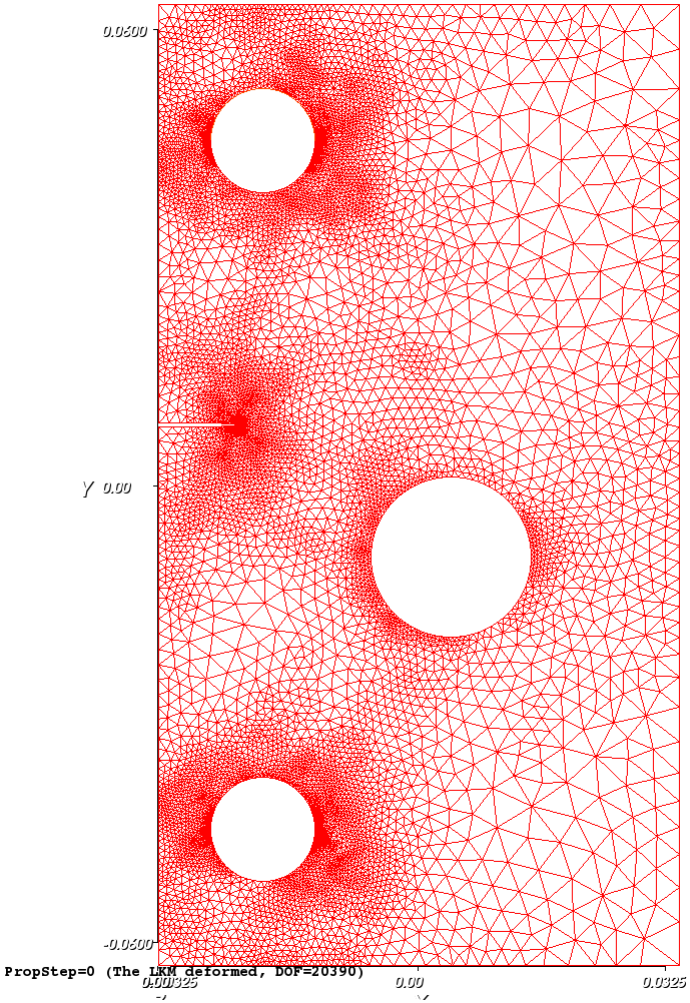
Adapt. Step 3
(20390 DOF)

Propagation angle

$$\theta_{\text{Prop}} = 1.525^{\circ}$$

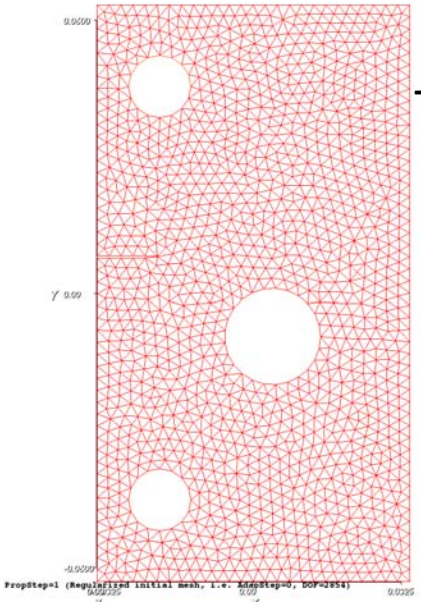
for the next
prop. step

Propagating crack opening



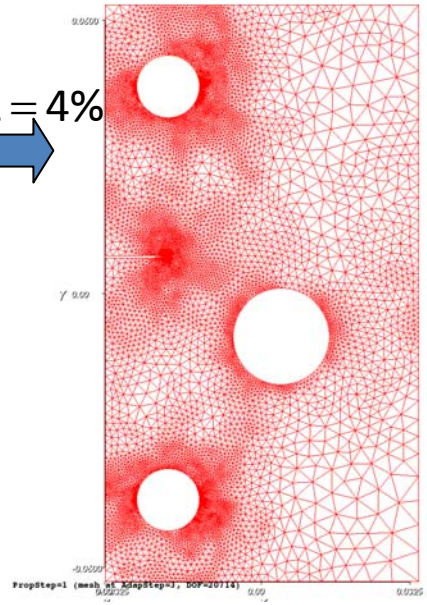
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 1



Initial mesh
(2854 DOF)

TOL = 4%



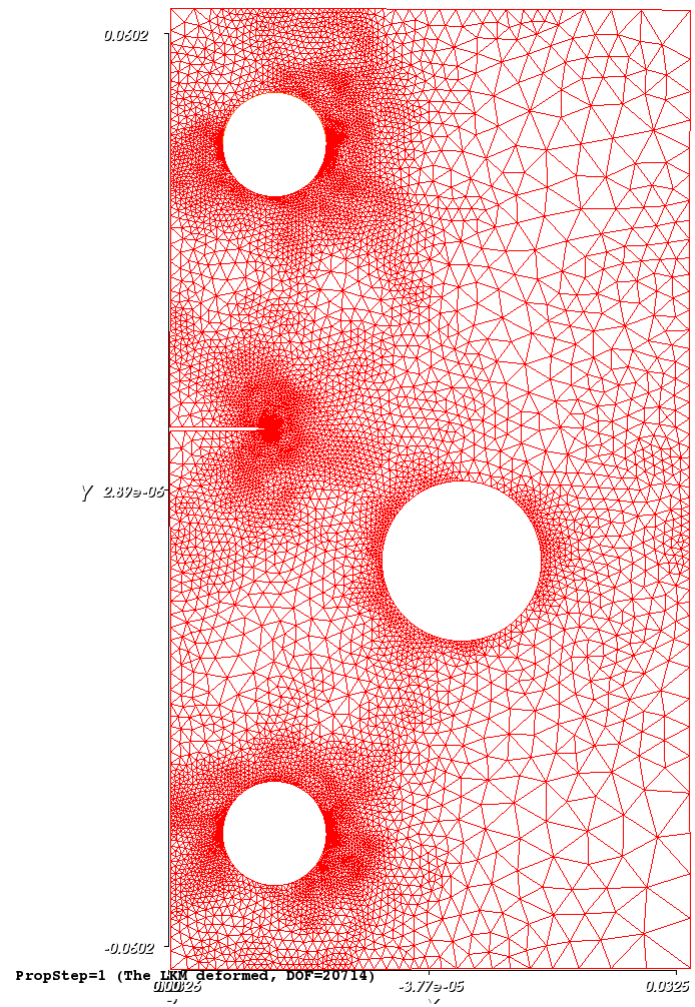
Adapt. Step 3
(20714 DOF)

Propagation angle

$$\theta_{\text{Prop}} = 0.875^{\circ}$$

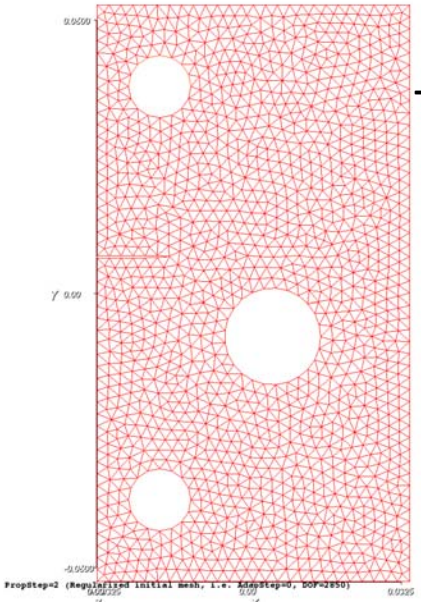
for the next
prop. step

Propagating crack opening



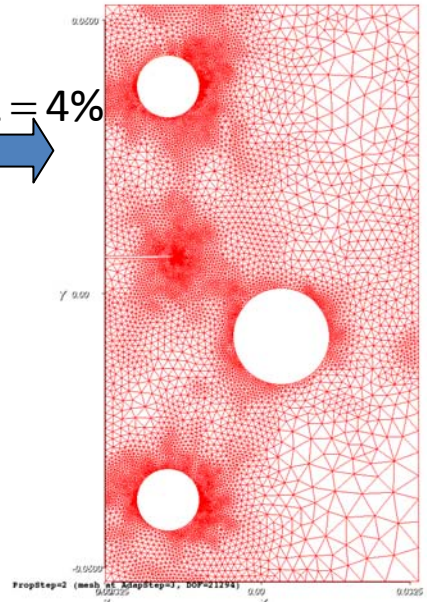
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 2



Initial mesh
(2850 DOF)

TOL = 4%



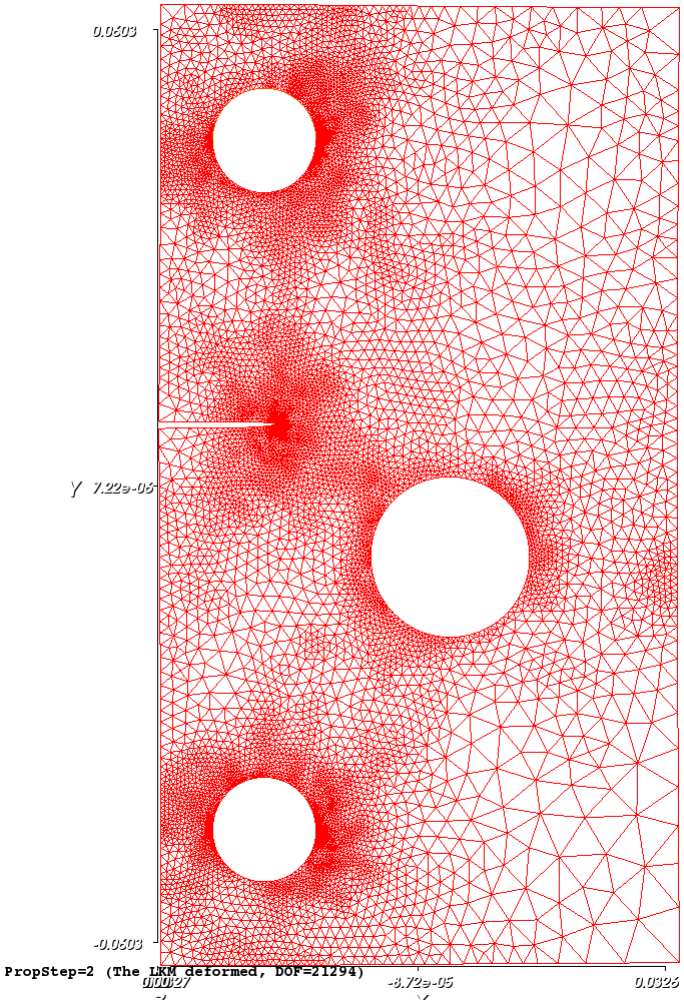
Adapt. Step 3
(21294 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -0.05^{\circ}$$

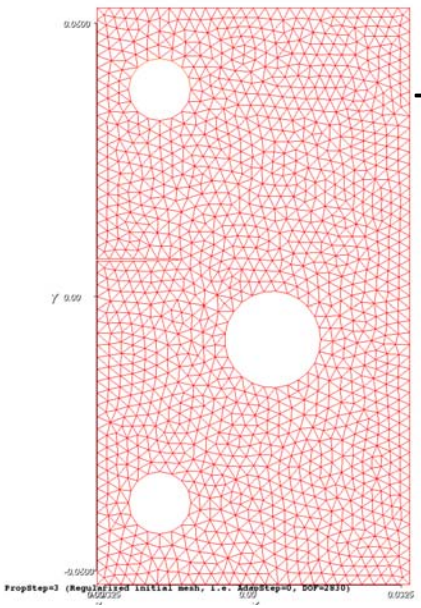
for the next
prop. step

Propagating crack opening



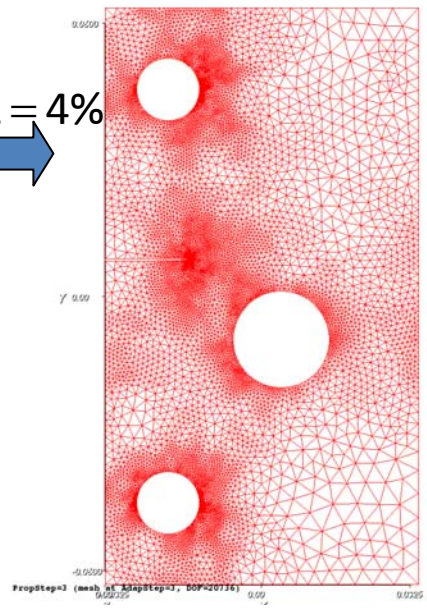
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 3



Initial mesh
(2830 DOF)

TOL = 4%



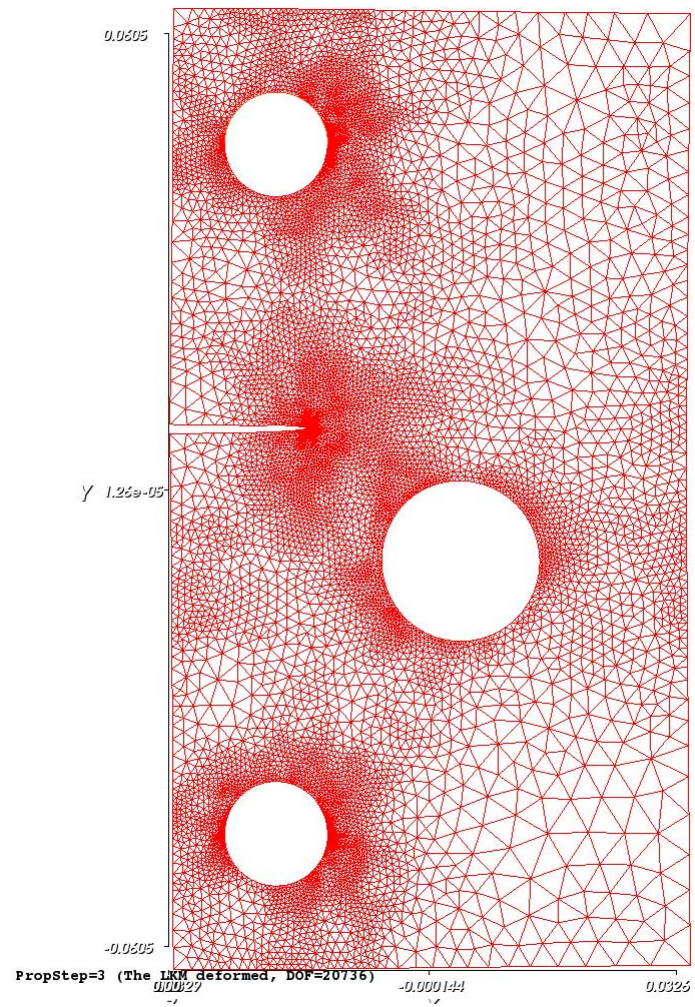
Adapt. Step 3
(20736 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -1.4^{\circ}$$

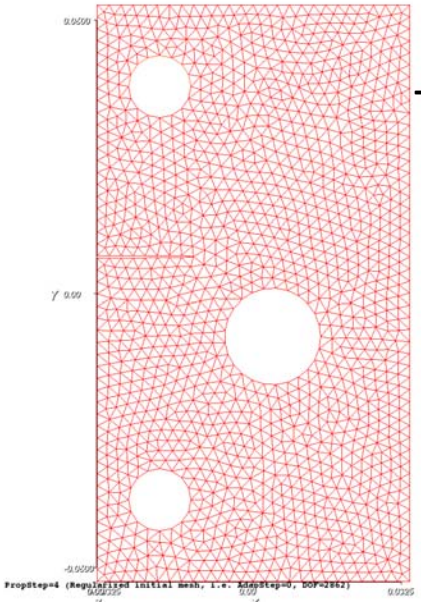
for the next
prop. step

Propagating crack opening



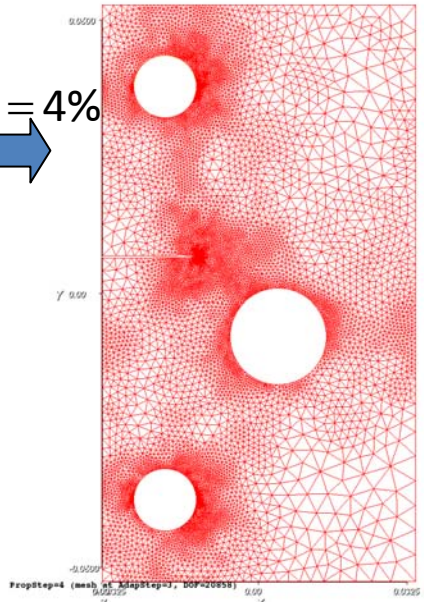
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 4



Initial mesh
(2862 DOF)

TOL = 4%



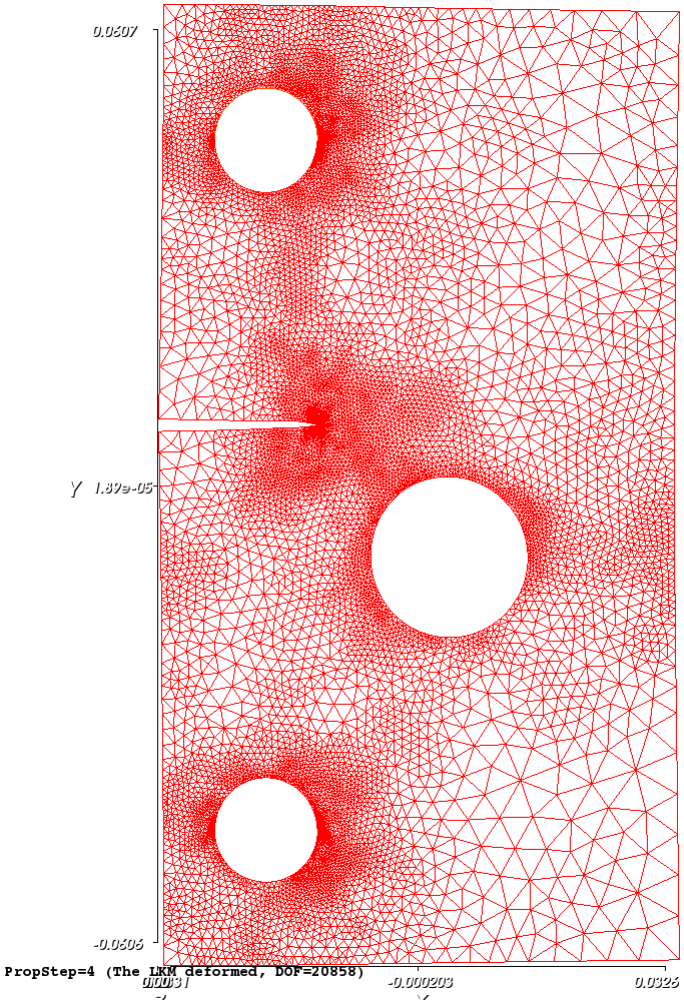
Adapt. Step 3
(20858 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -3.05^{\circ}$$

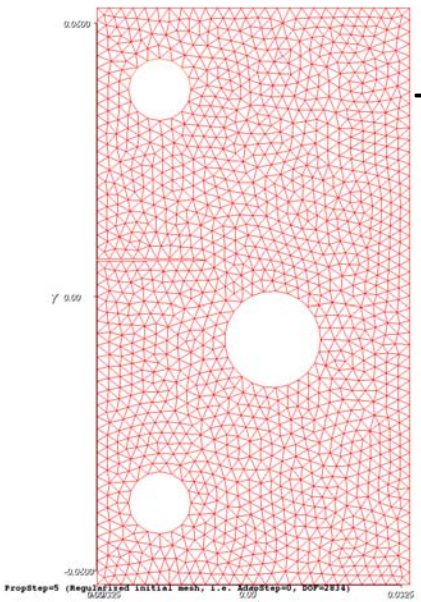
for the next
prop. step

Propagating crack opening



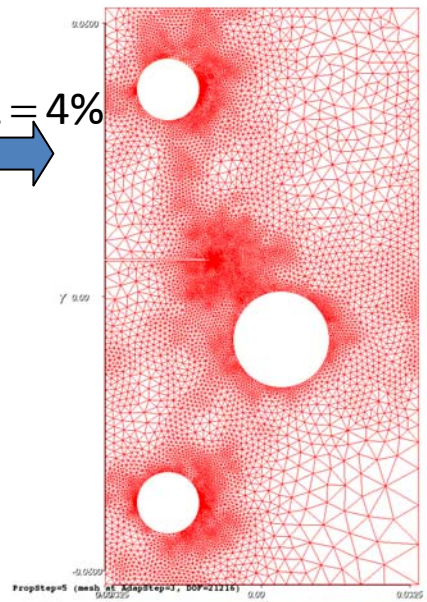
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 5



Initial mesh
(2834 DOF)

TOL = 4%



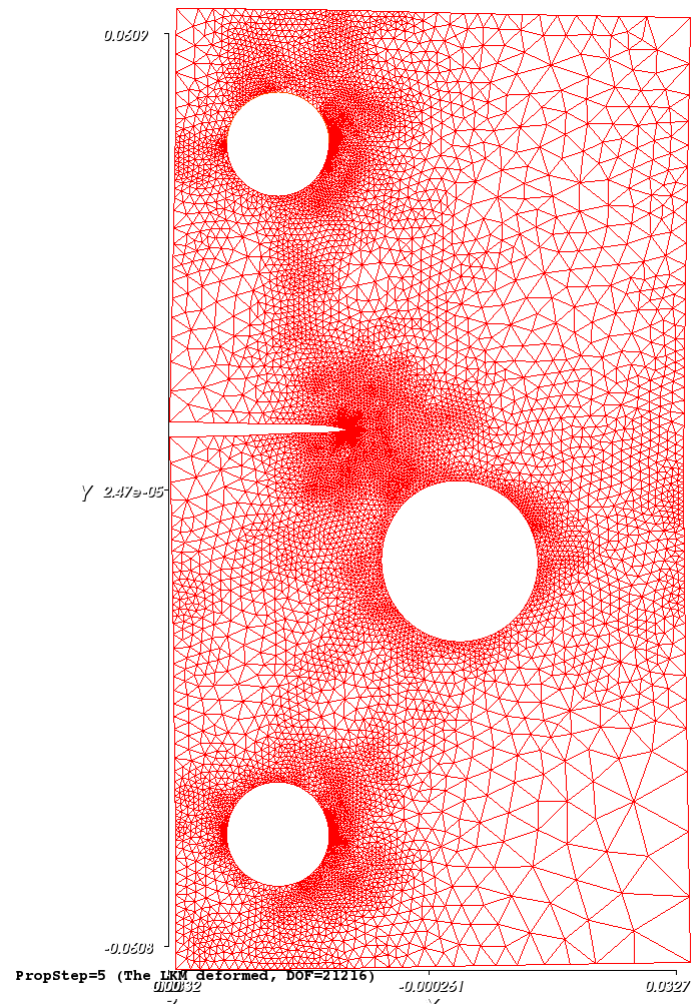
Adapt. Step 3
(21216 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -5.34^{\circ}$$

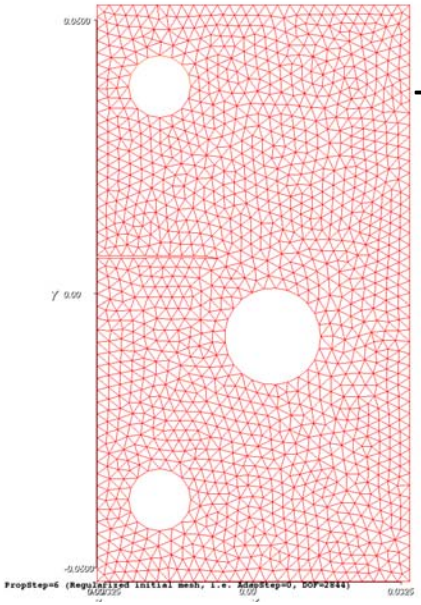
for the next
prop. step

Propagating crack opening



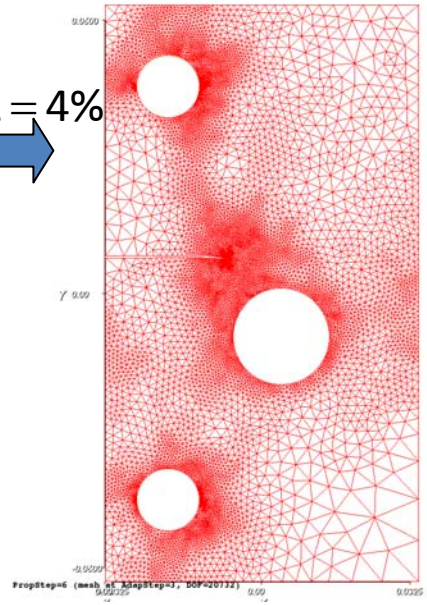
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 6



Initial mesh
(2844 DOF)

TOL = 4%



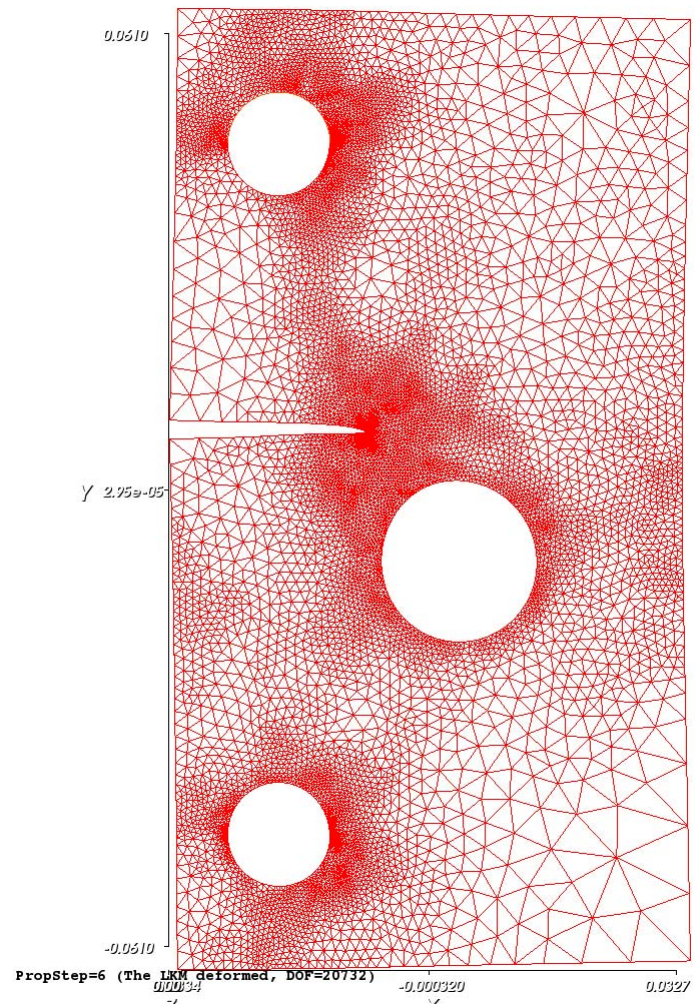
Adapt. Step 3
(20732 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -8.175^{\circ}$$

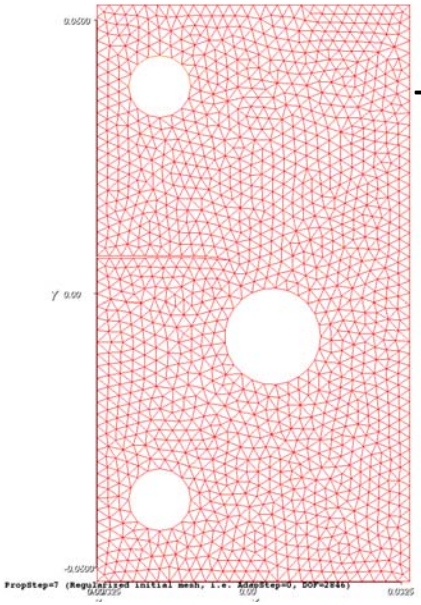
for the next
prop. step

Propagating crack opening



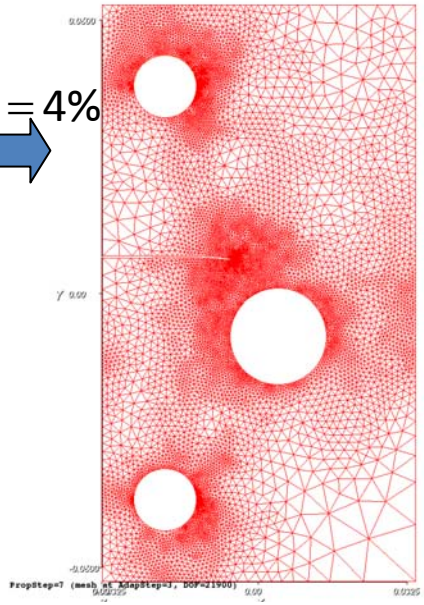
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 7



Initial mesh
(2846 DOF)

TOL = 4%



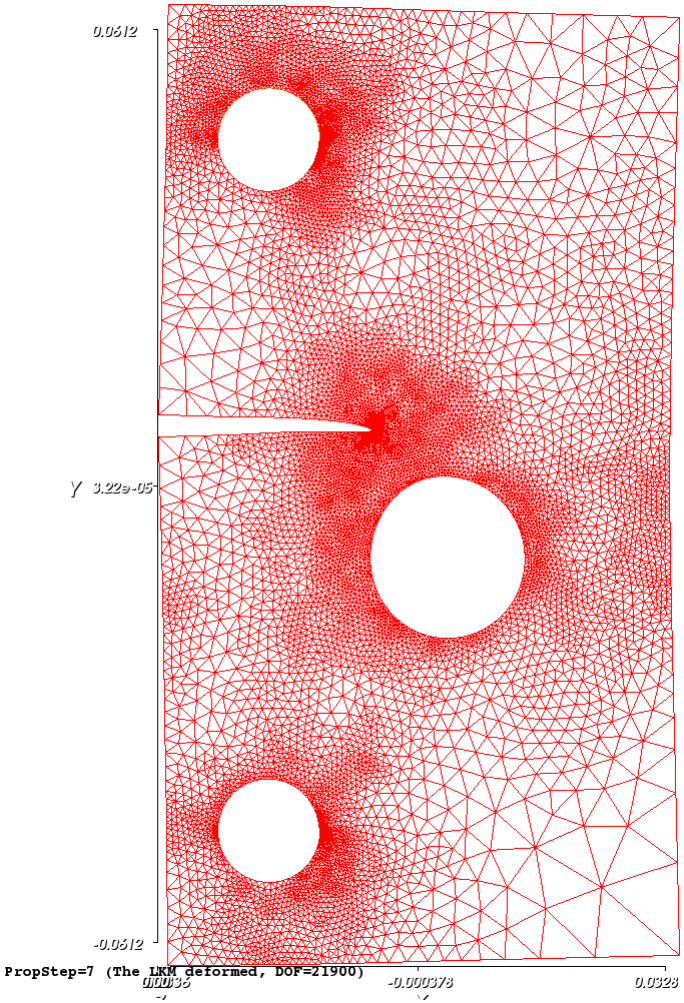
Adapt. Step 3
(21900 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -11.81^{\circ}$$

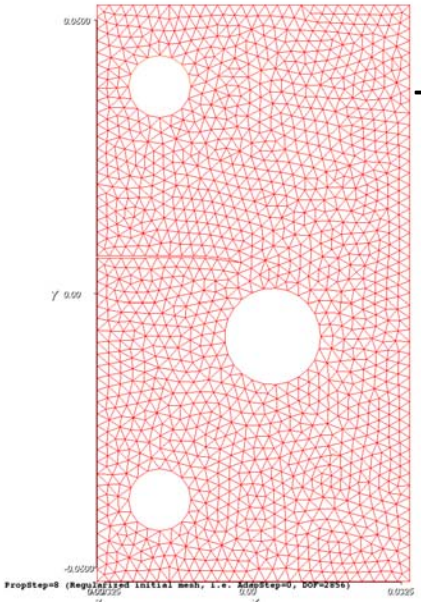
for the next
prop. step

Propagating crack opening



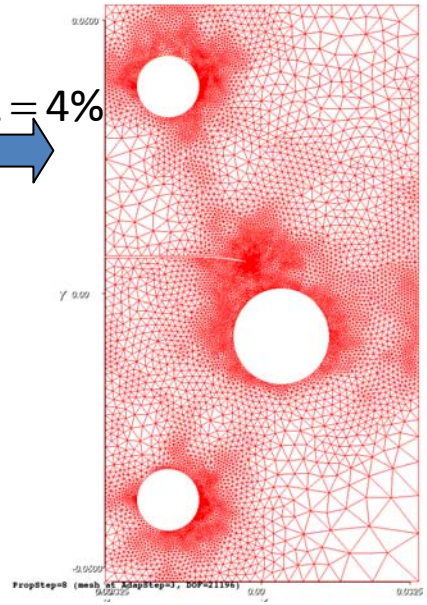
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 8



Initial mesh
(2856 DOF)

TOL = 4%



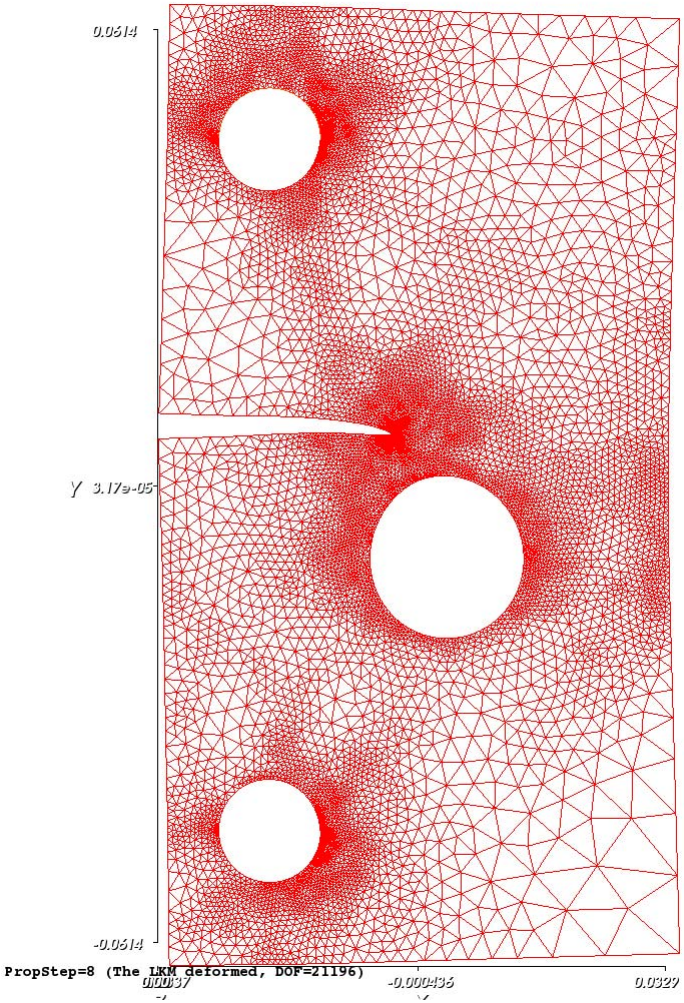
Adapt. Step 3
(21196 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -16.425^{\circ}$$

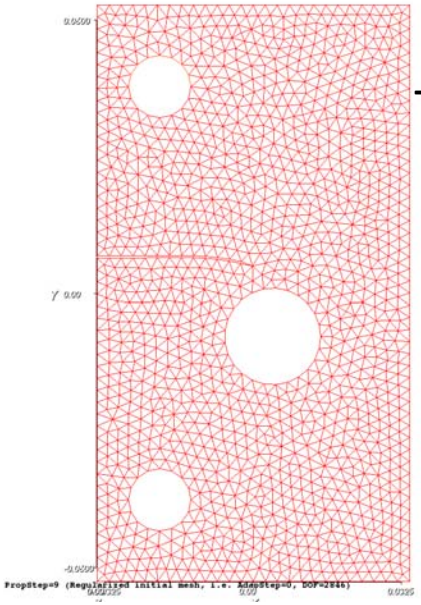
for the next
prop. step

Propagating crack opening



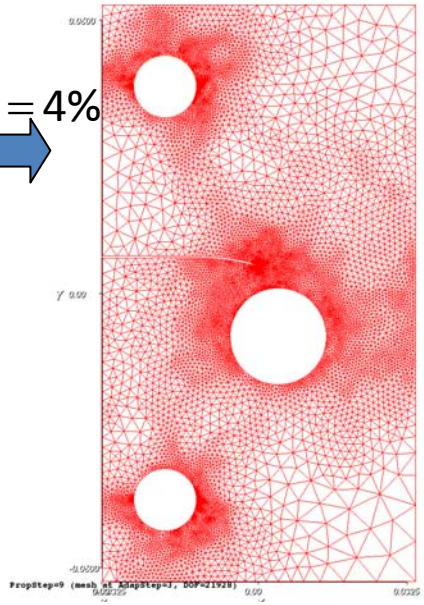
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 9



Initial mesh
(2846 DOF)

TOL = 4%



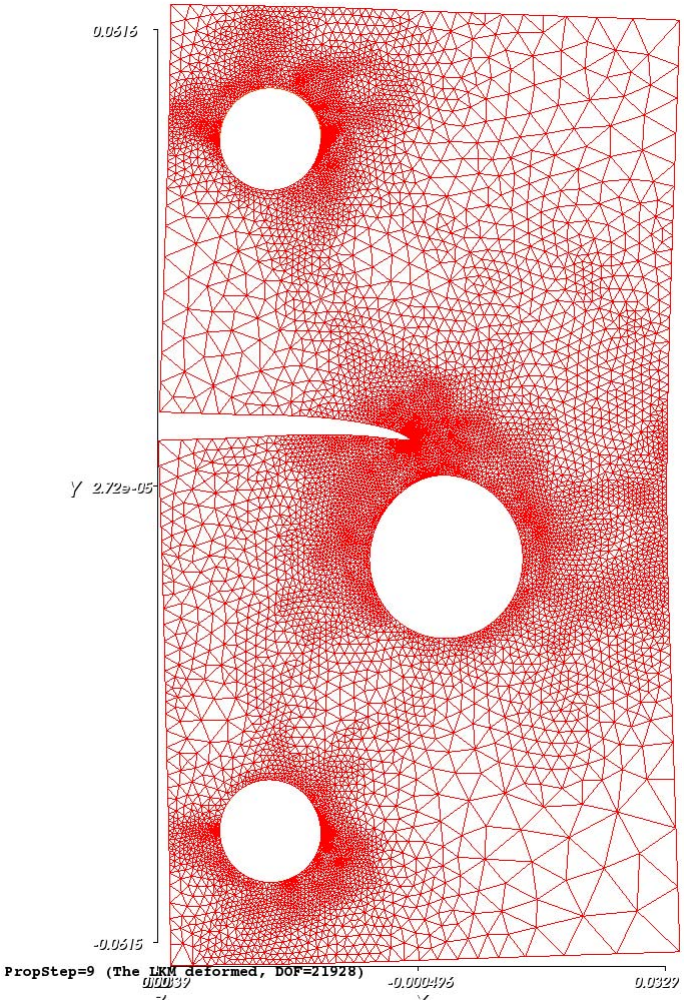
Adapt. Step 3
(21928 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -22.46^{\circ}$$

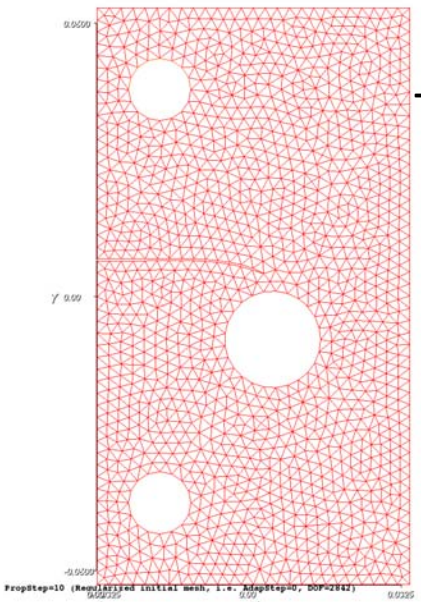
for the next
prop. step

Propagating crack opening



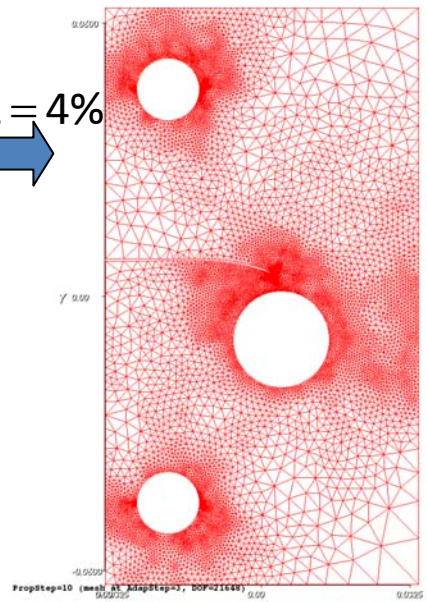
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 10



Initial mesh
(2842 DOF)

TOL = 4%



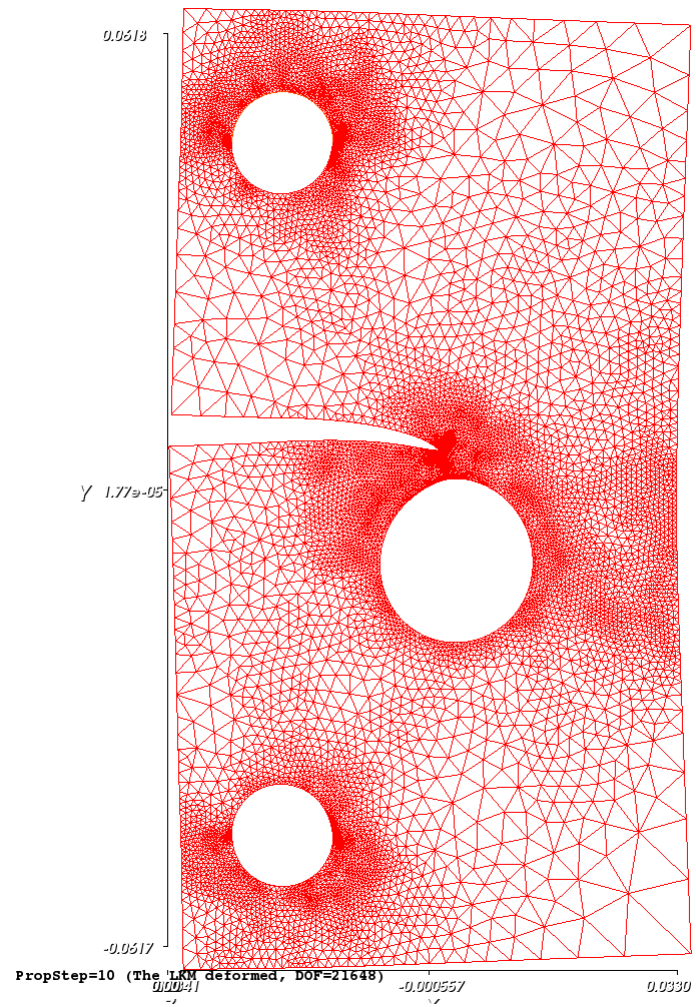
Adapt. Step 3
(21648 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -30.675^{\circ}$$

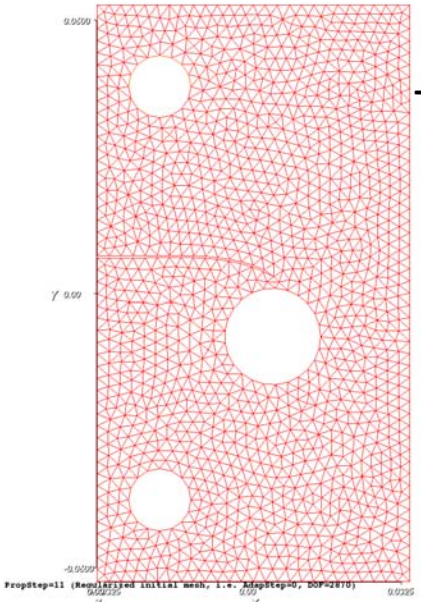
for the next
prop. step

Propagating crack opening



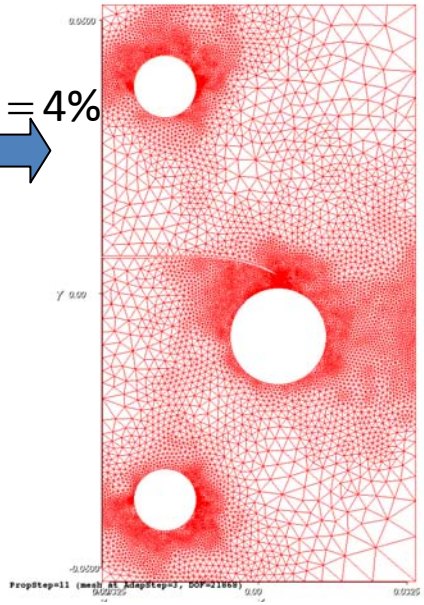
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 11



Initial mesh
(2870 DOF)

TOL = 4%



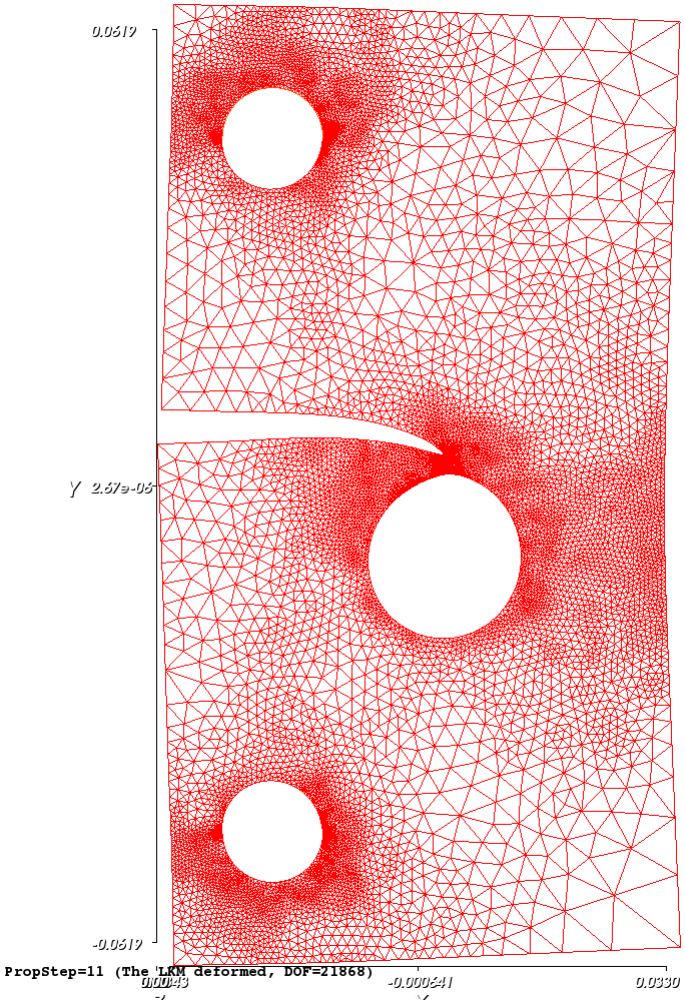
Adapt. Step 3
(21868 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -41.85^{\circ}$$

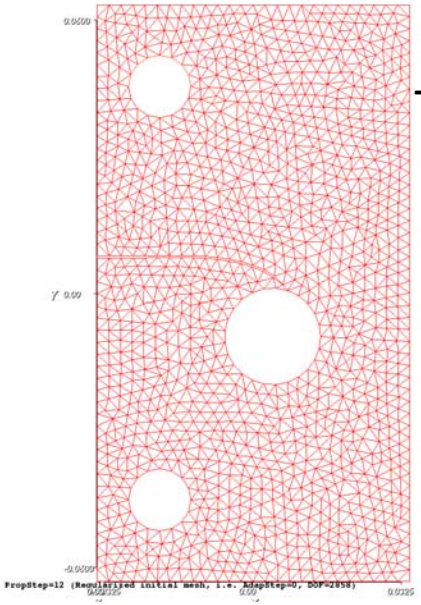
for the next
prop. step

Propagating crack opening



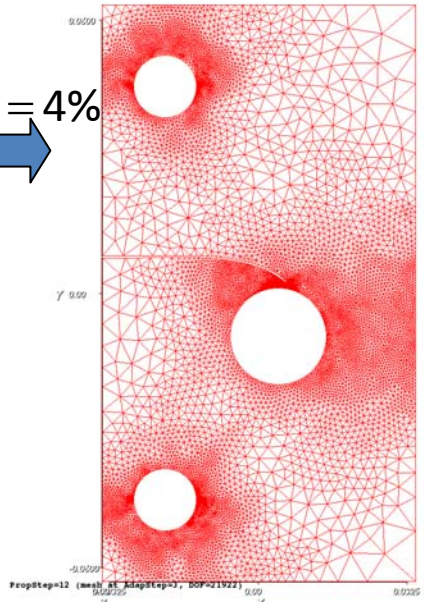
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 12



Initial mesh
(2858 DOF)

TOL = 4%



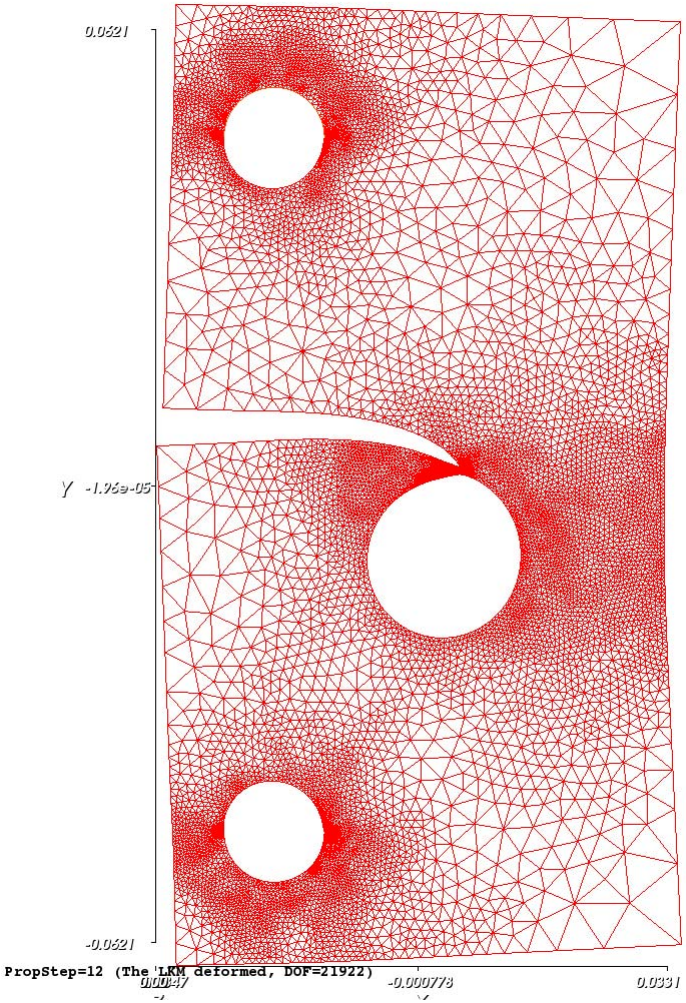
Adapt. Step 3
(21922 DOF)

Propagation angle

$$\theta_{\text{Prop}} = -54.82^{\circ}$$

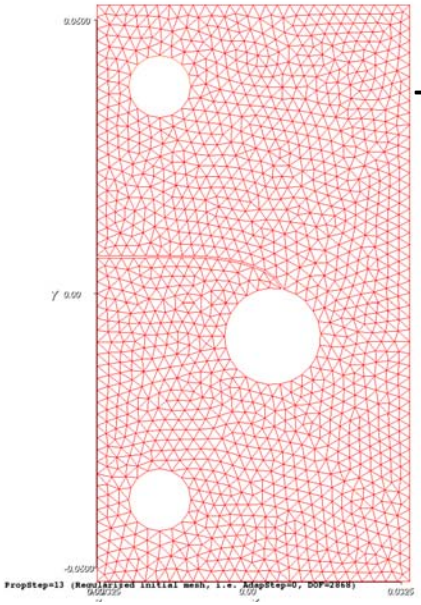
for the next
prop. step

Propagating crack opening



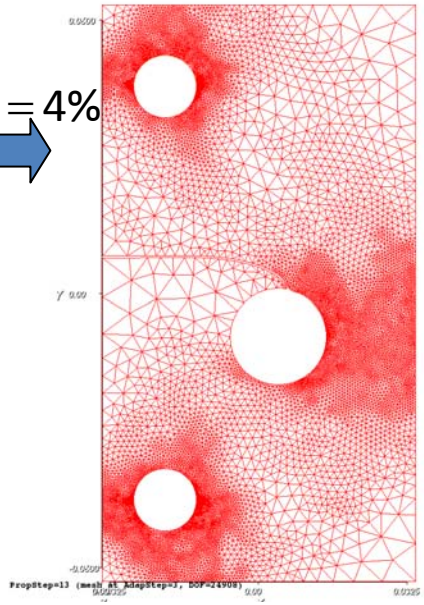
Application I:
Error-controlled crack propagation (in 2D)

Prop. Step 13
(we stop here)



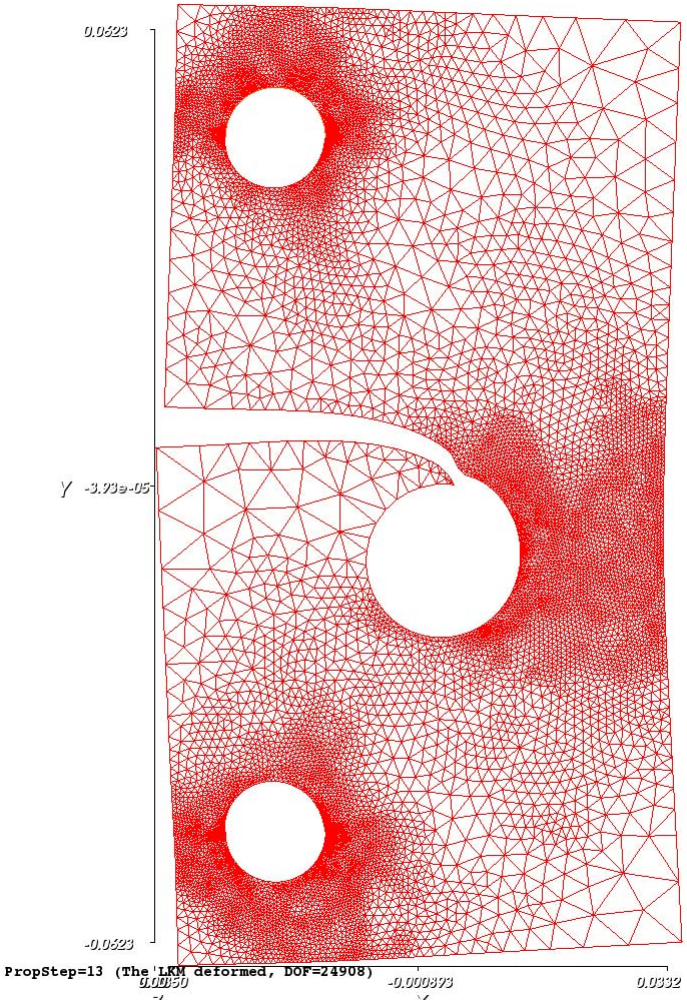
Initial mesh
(2868 DOF)

TOL = 4%



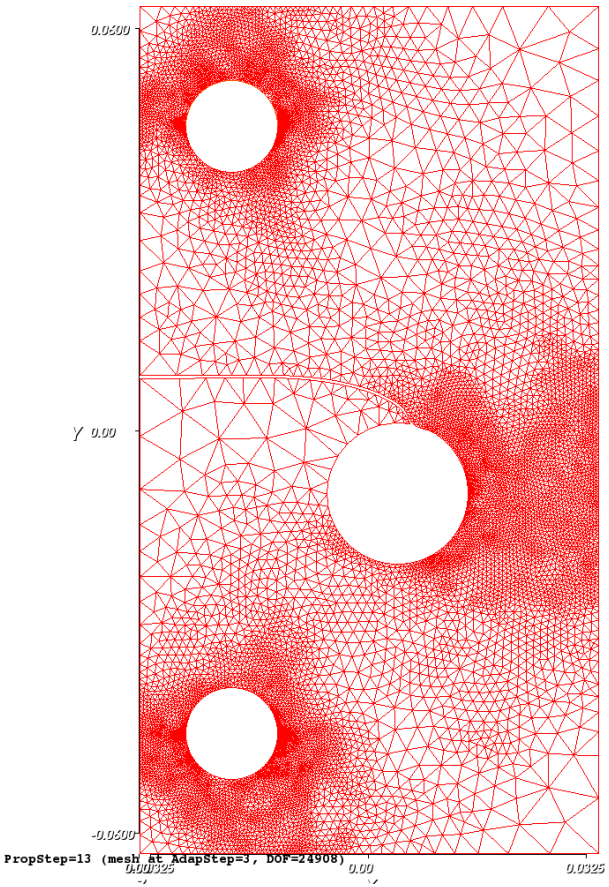
Adapt. Step 3
(24908 DOF)

Final failure



Application I:
Error-controlled crack propagation (in 2D)

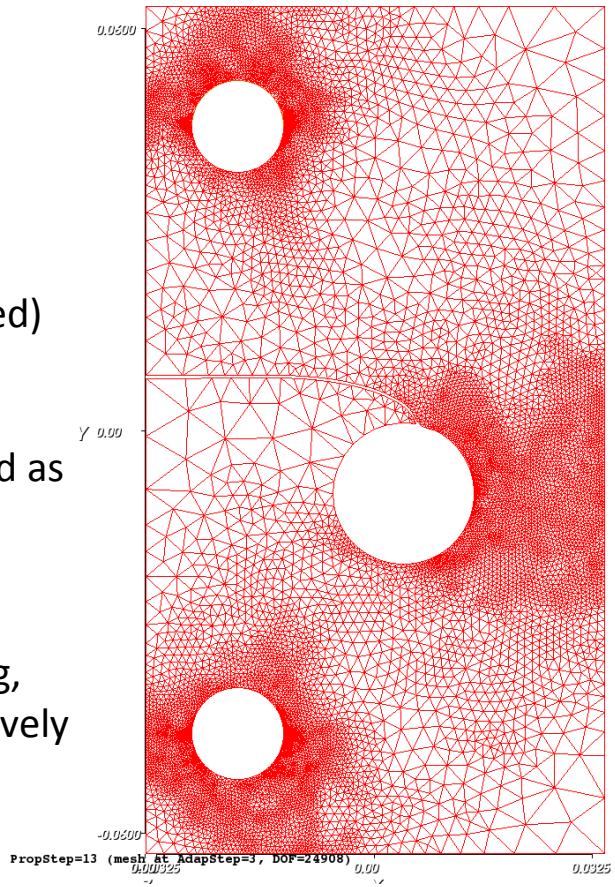
compare



Application I:
Error-controlled crack propagation (in 2D)

Note,

- **low-order elements** were used (the costs are really affordable)
- **a standard** (not purpose-oriented) **comp. code** [1] was used
- a progressing crack was modeled as a **wedge** (not a slit, as e.g. in XFEM/GFEM)
- **TOL:=4%** for adaptive remeshing, and crack **increment** were relatively **large**.



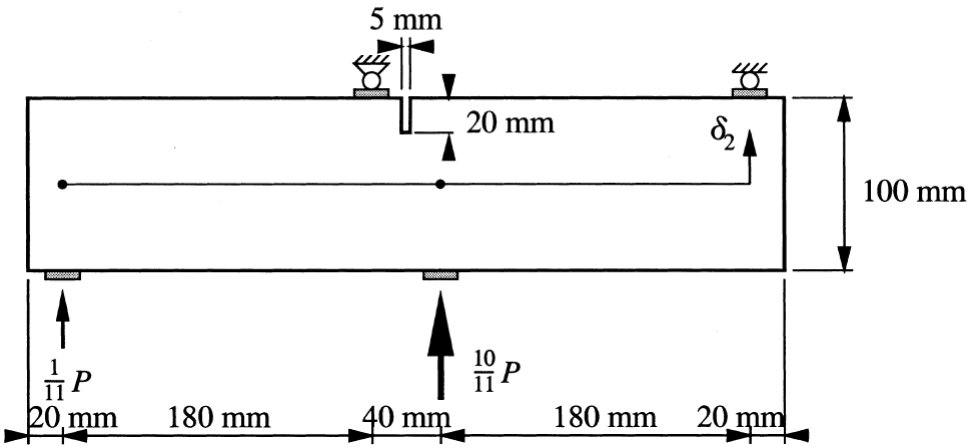
Yet, the results are accurate enough, what we mainly attribute to mesh adaptivity, driven by the new error estimator proposed.

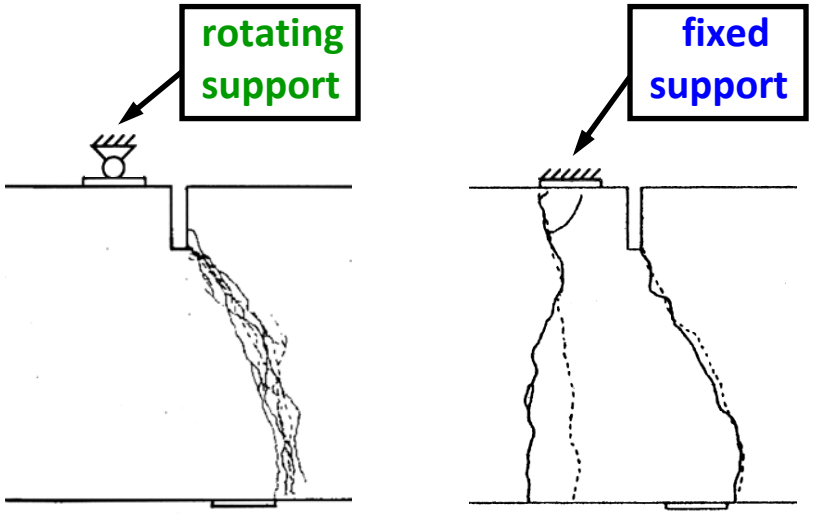
[1] ***FreeFem++* software** (F.Hecht, Laboratoire J.-L. Lions, Université P. et M. Curie, Paris)

Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:





Also in [2] for the "small" specimen

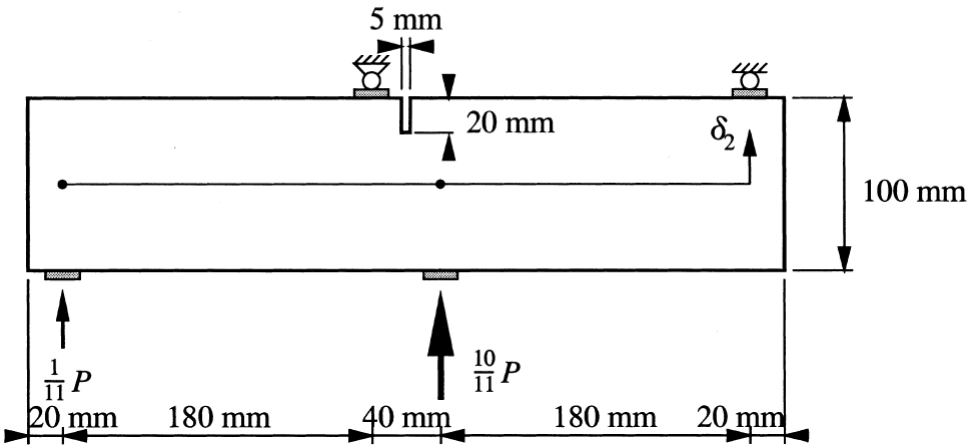
[1] E. Schlangen Experimental and numerical analysis of fracture processes in concrete, *PhD Thesis*, Delft University of Technology, 1993.

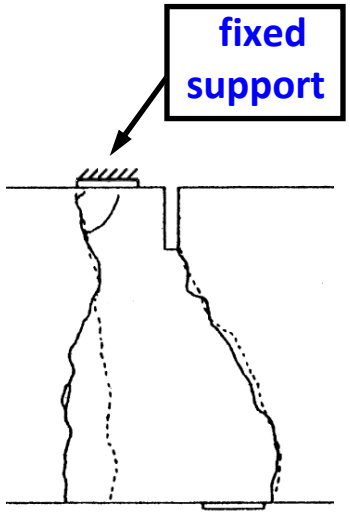
[2] A. Carpinteri et al. Is mode II fracture energy a real material property? *Computers and structures* 1993; 48(3):397-413.

Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:





We are interested in accurate modeling of this particular case:

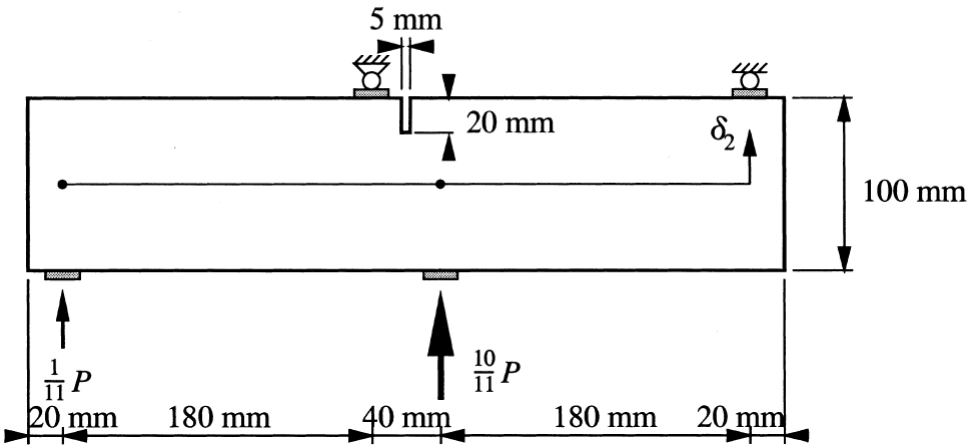
the two non-symmetric cracks nucleate and evolve

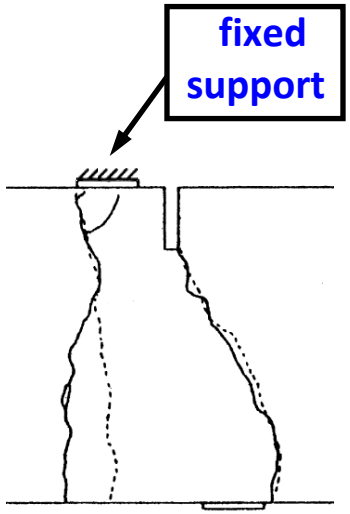
(the left one is seemingly "counter intuitive")

Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

Experimental results from [1]:





We adopt the comput. model from [2]:

$$\boldsymbol{\sigma} = (1 - D(\bar{\varepsilon}_{eq}))\,\mathbb{C} : \varepsilon \quad \text{in} \quad \Omega$$

$$\begin{cases} -c\nabla^2 \bar{\varepsilon}_{eq} + \bar{\varepsilon}_{eq} = \varepsilon_{eq}(\varepsilon) & \text{in} \quad \Omega \\ \nabla \bar{\varepsilon}_{eq} \cdot \boldsymbol{n} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

- specific definition of equivalent strain

$$\varepsilon_{eq} := f(k, J_1(\varepsilon), J_2(\varepsilon))$$

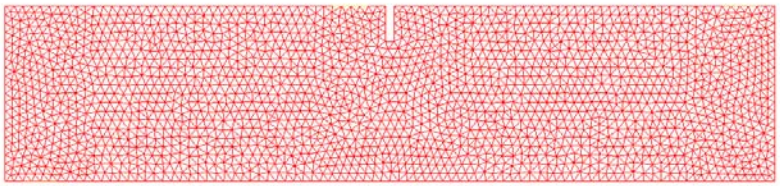
- calibrated: k, c and κ_0, α, β in D

[2] M.G.D. Geers, R. de Borst, R.H.J. Peerlings Damage and crack modeling in single-edge and double-edge notched concrete beams, *Engineering Fracture Mechanics* 2000; **65**:247-261

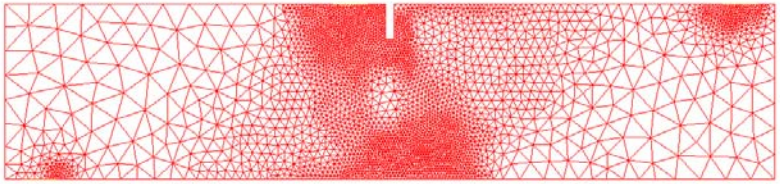
Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

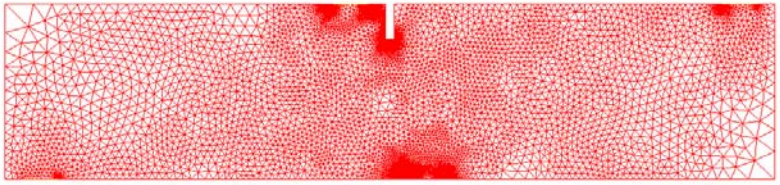
We first adapt mesh
on the **pre-damaged** stage…



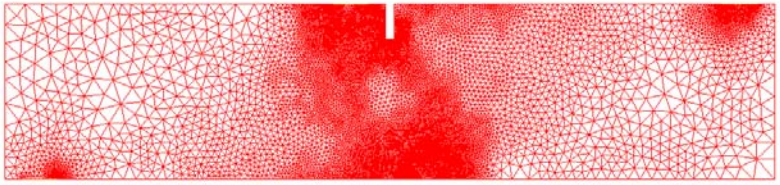
4026 DOF



6714 DOF



12712 DOF

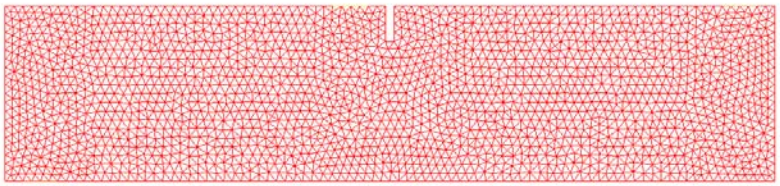


26986 DOF

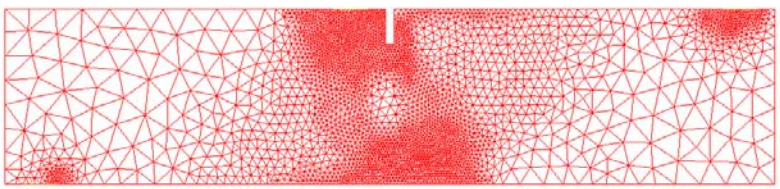
Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

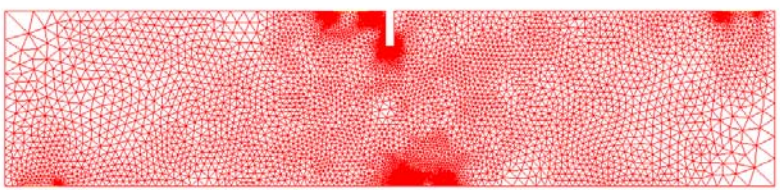
We first adapt mesh
on the **pre-damaged** stage…



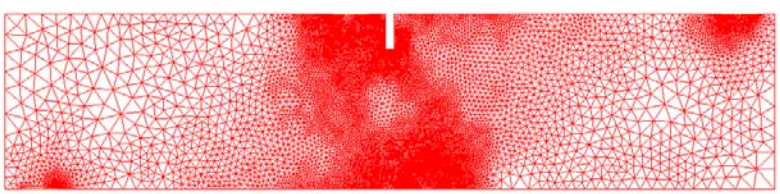
4026 DOF



6714 DOF



12712 DOF



26986 DOF

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…when the above system
is naturally decoupled.

Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

We first adapt mesh
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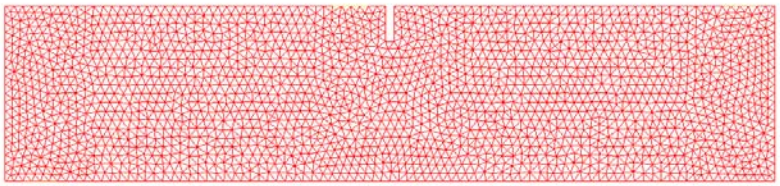
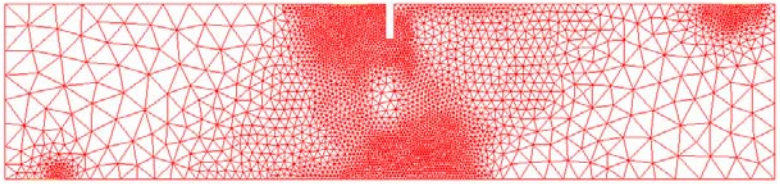
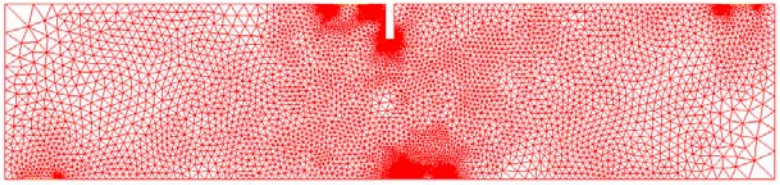


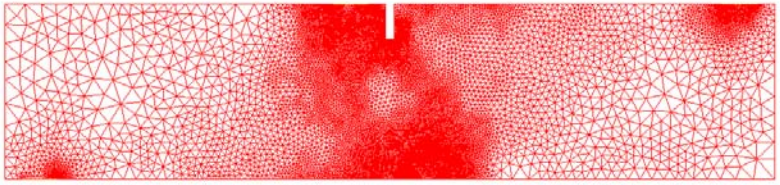
4026 DOF



6714 DOF



12712 DOF



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…when the above system
is naturally decoupled.

Hence, we can use our
error estimator

$$\|\mathbf{e}\|_{\Omega} \le \frac{\tilde{c}_P \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE}\lambda}} \left(\sum_T \eta_{T,New}^2 \right)^{1/2} =: UB_{New}$$

TOL = 1 %

Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

We first adapt mesh
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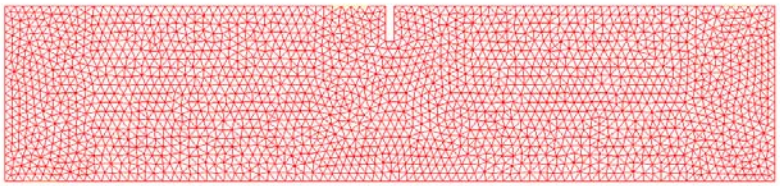

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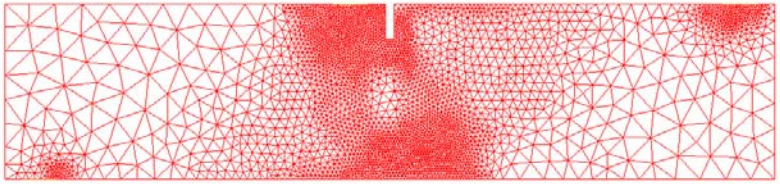

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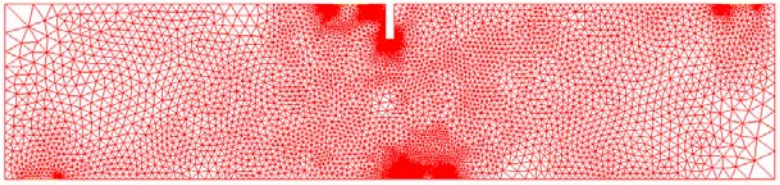
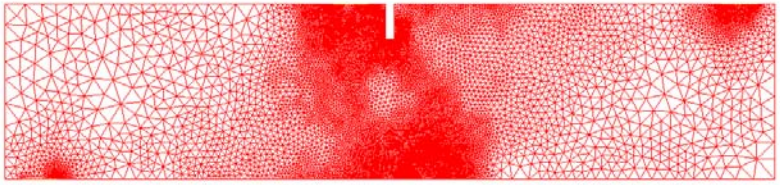
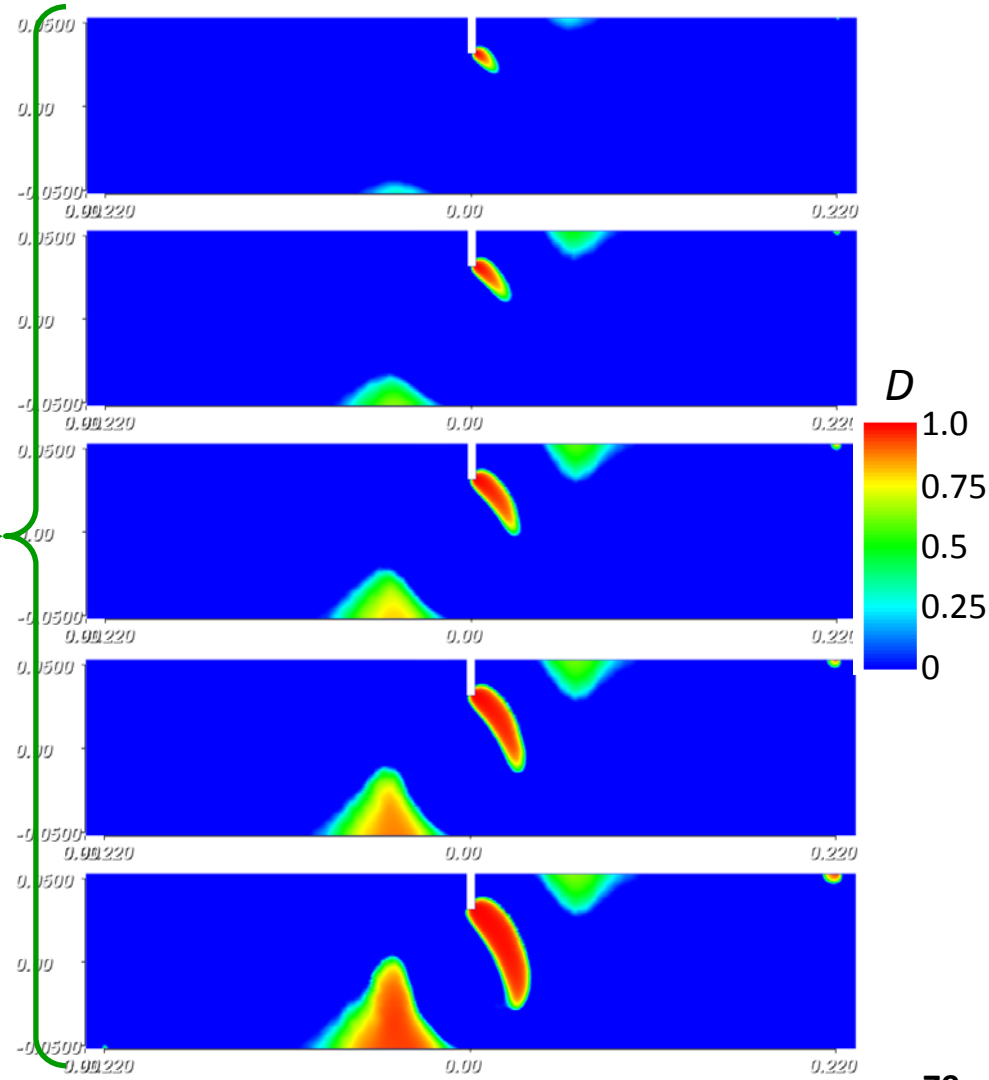


12712 DOF



26986 DOF

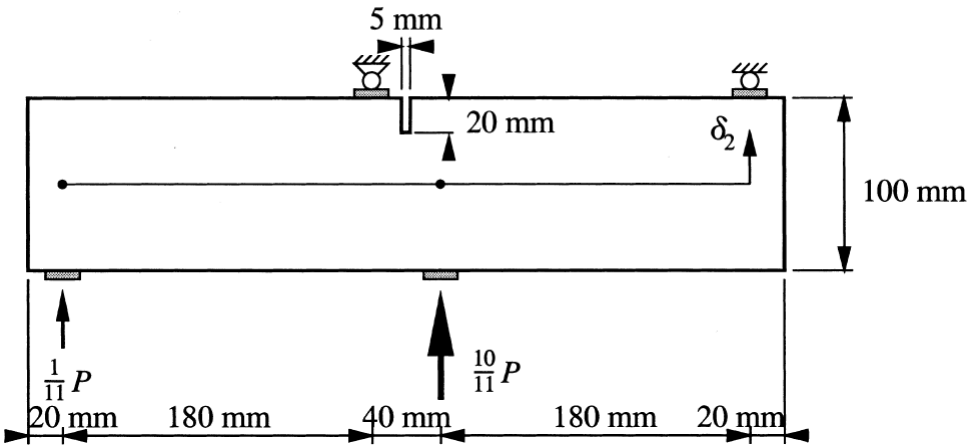
… and then compute the damage *D* evolution

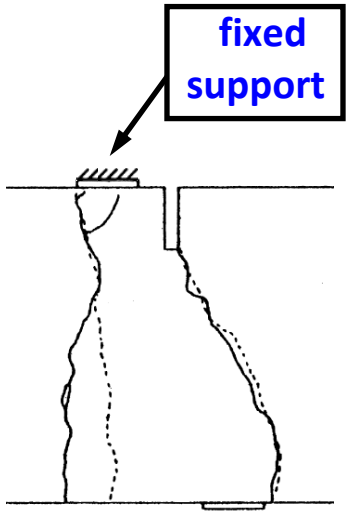


Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

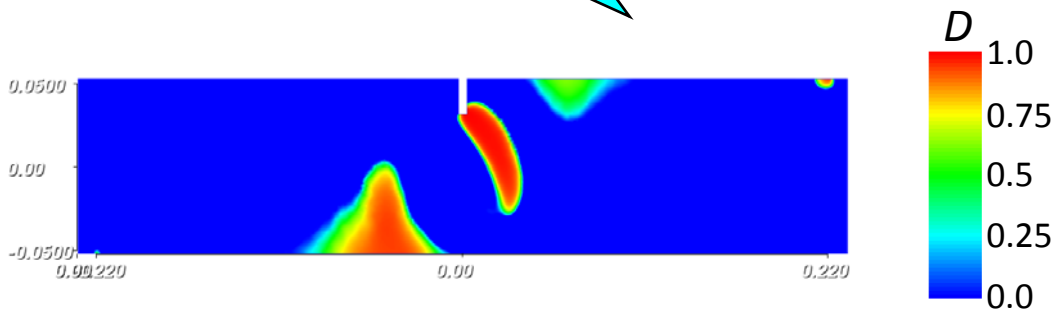
Experimental results from [1]:





compare

Damage state at a certain loading [2]:



[2] T. Gerasimov, E. Stein and P. Wriggers (2012) Fully adaptive FEM/SFM for 2D crack propagation and damage analysis using a new simple constant-free explicit error estimator, *in preparation*.

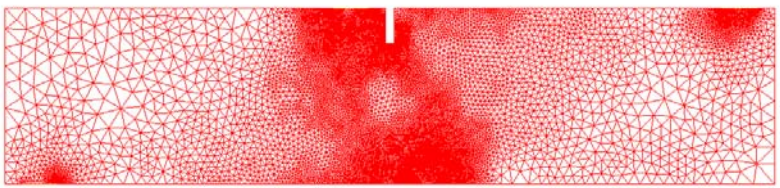
Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

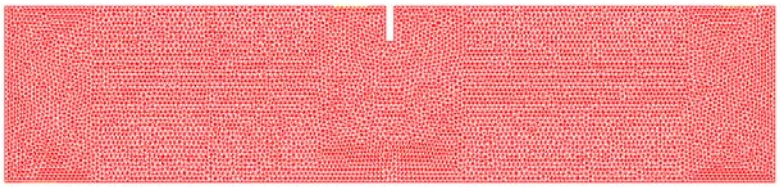
Adapted mesh

Uniform mesh

for damage evolution computations

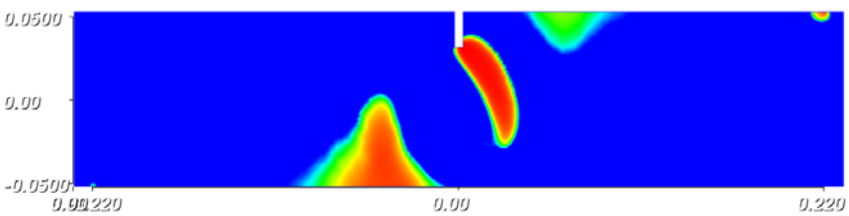


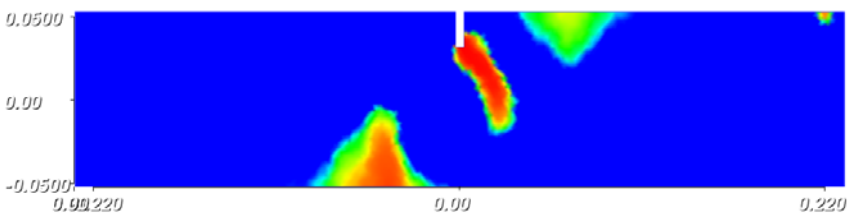
26986 DOF

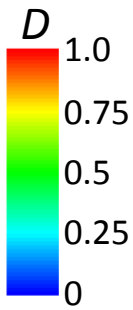


20684 DOF

Damage state for the same loading







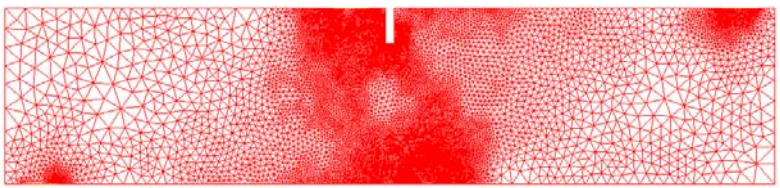
Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

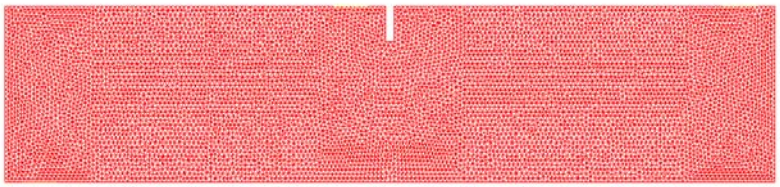
Adapted mesh

Uniform mesh

for damage evolution computations

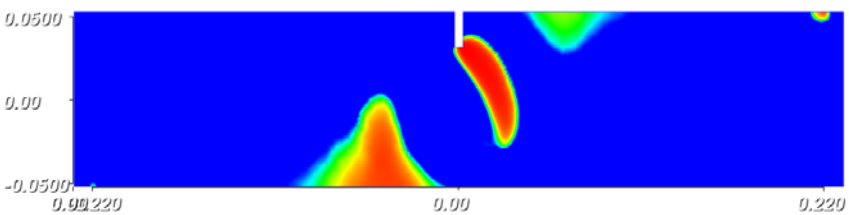


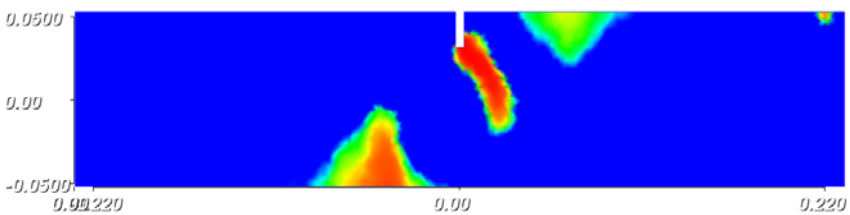
26986 DOF

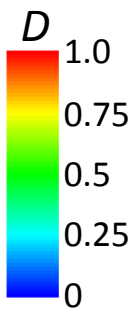


20684 DOF

Damage state for the same loading







Transformation of a **damage zone** into an **equivalent crack**, [1]

[1] J. Mazars, G. Pijaudier-Cabot From damage to fracture mechanics and conversely: a combined approach, *Int J Solid Struct* 1996; **33**:3327-3342.

Application II:

Quasi-error-controlled damage analysis for a failure of SEN concrete beam

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Uniform mesh

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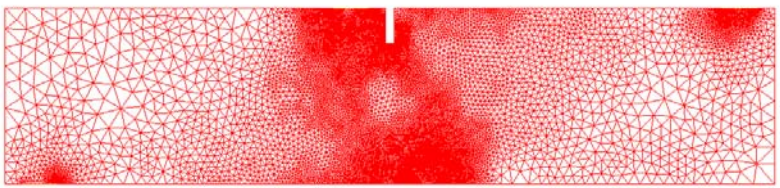
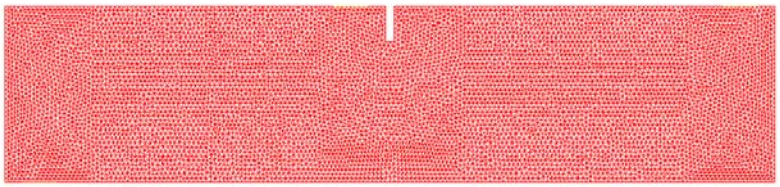
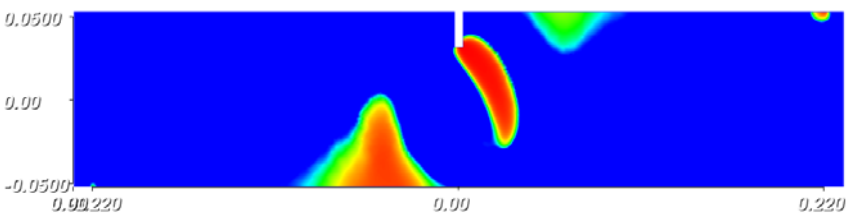


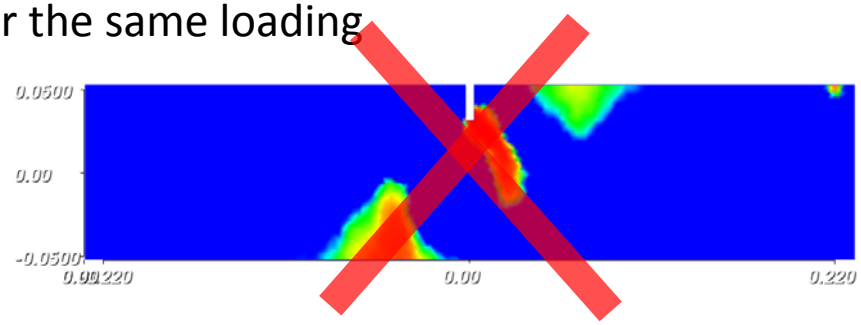
26986 DOF

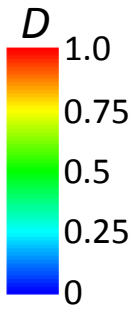


20684 DOF

Damage state for the same loading



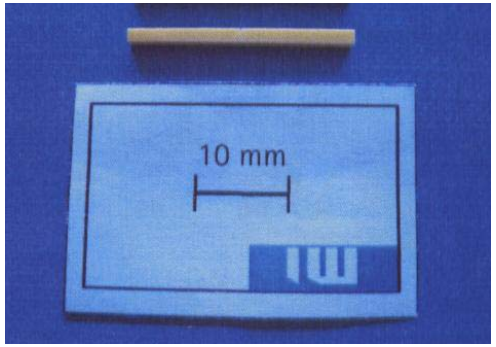




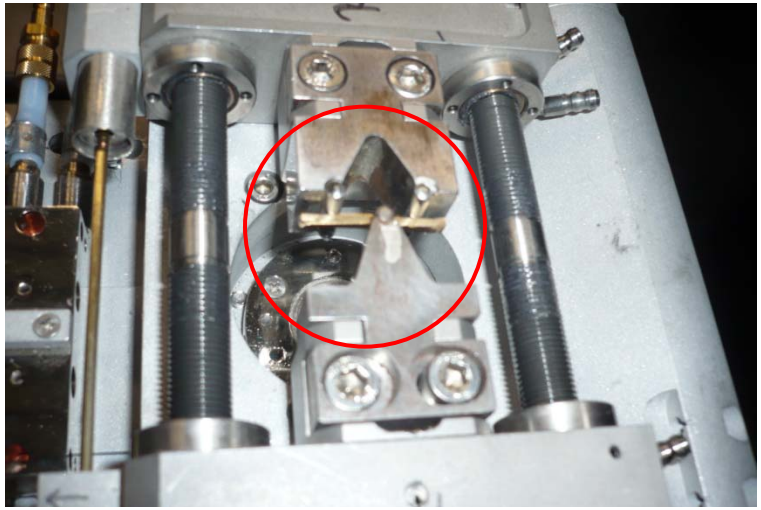
Transformation of a **damage zone** into an **equivalent crack**, [1]

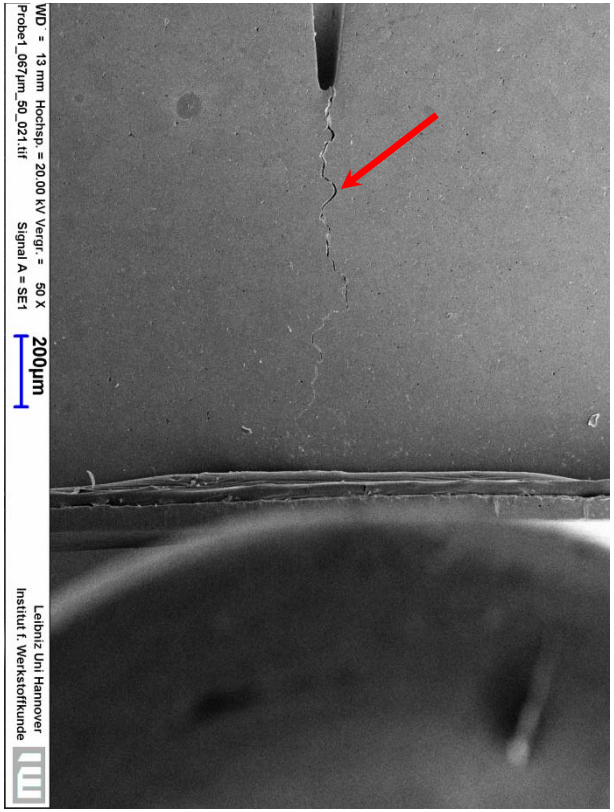
[1] J. Mazars, G. Pijaudier-Cabot From damage to fracture mechanics and conversely: a combined approach, *Int J Solid Struct* 1996; **33**:3327-3342.

Application III:
Micro-crack initiation in a ceramic specimen



Magnesium-stabilized Zirconia dioxide ($Mg\text{-}ZrO_2$) ceramic beam specimen of size 25x2x2 (dim. in mm) with a micro-notch of length ~330 µm under 3-point bending test

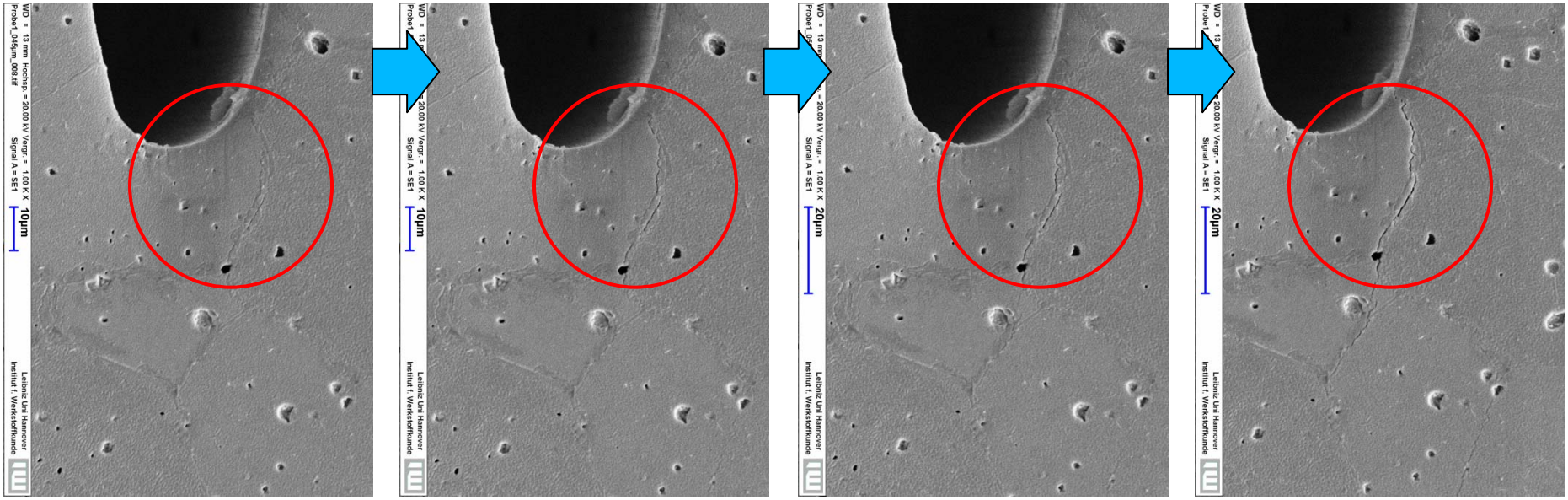




Micro-crack, stemming from the notch tip, shortly before rapture

Application III:

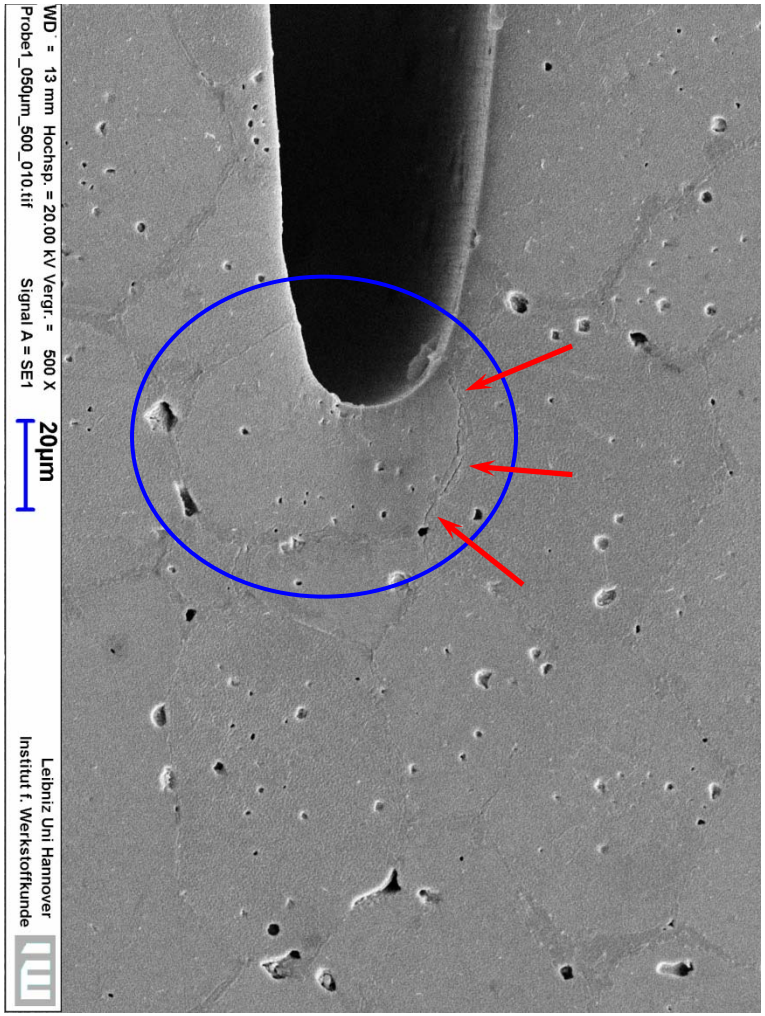
Micro-crack initiation in a ceramic specimen (first attempt)



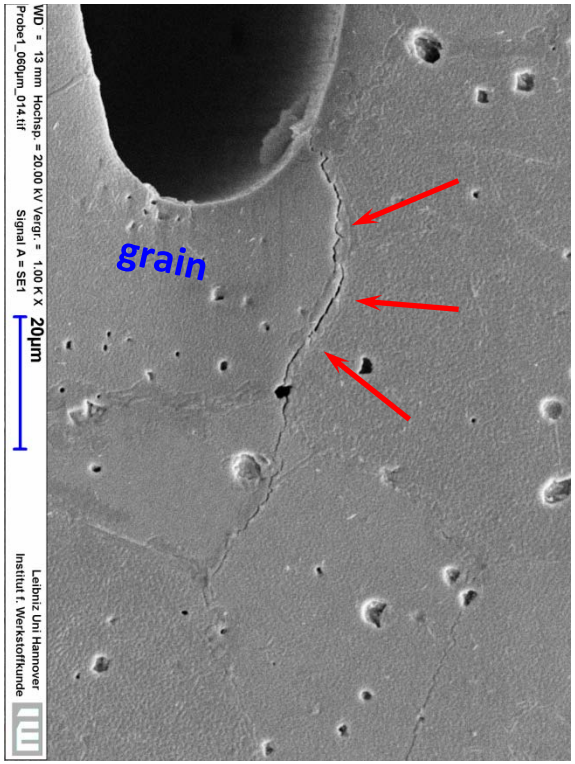
Initiation and development of a micro-crack

Application III:

Micro-crack initiation in a ceramic specimen (first attempt)



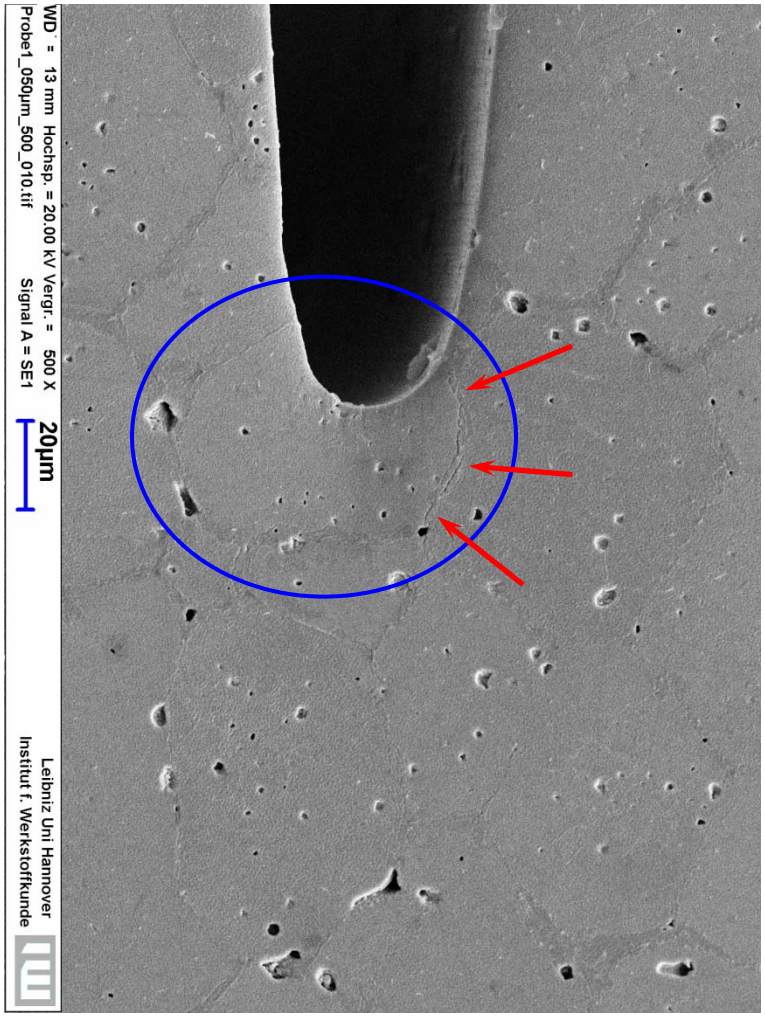
Grain (enclosed)



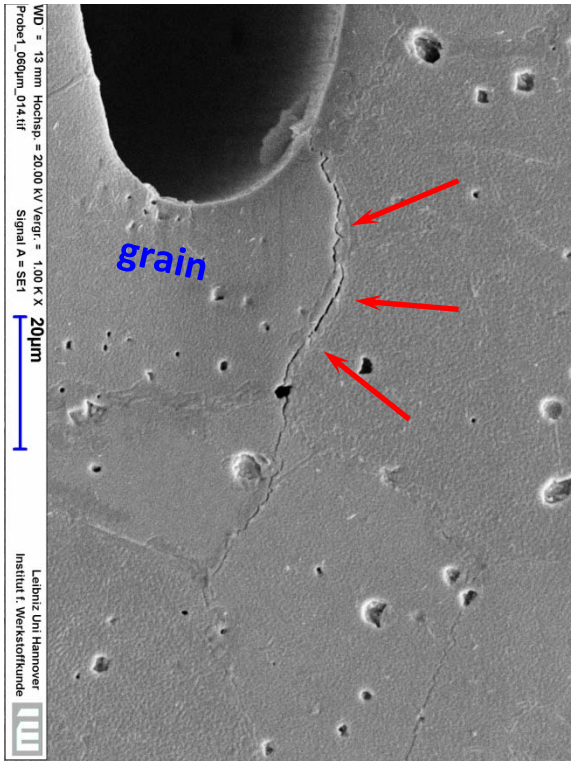
Micro-crack nucleates *inside* of the specimen, namely, on the **grain boundary** and develops along it

Application III:

Micro-crack initiation in a ceramic specimen (first attempt)



Grain (enclosed)



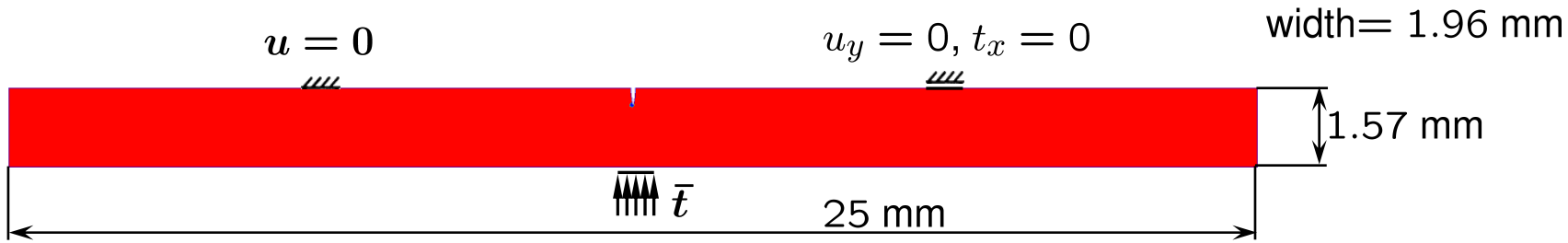
Micro-crack nucleates *inside* of the specimen, namely, on the **grain boundary** and develops along it

We will model this phenomena by using the damage model

Application III:

Micro-crack initiation in a ceramic specimen (first attempt)

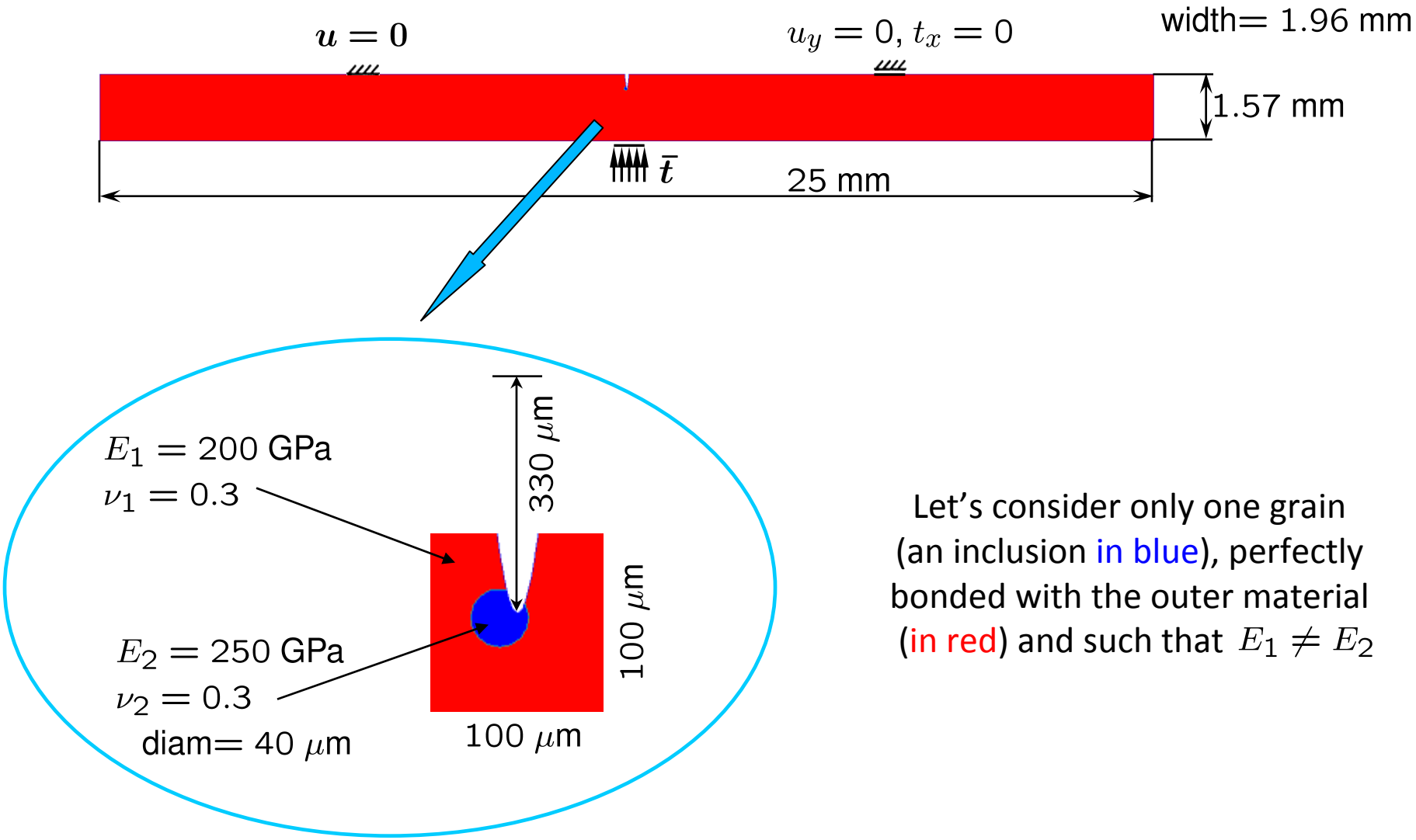
Our setup:



Application III:

Micro-crack initiation in a ceramic specimen (first attempt)

Our setup:

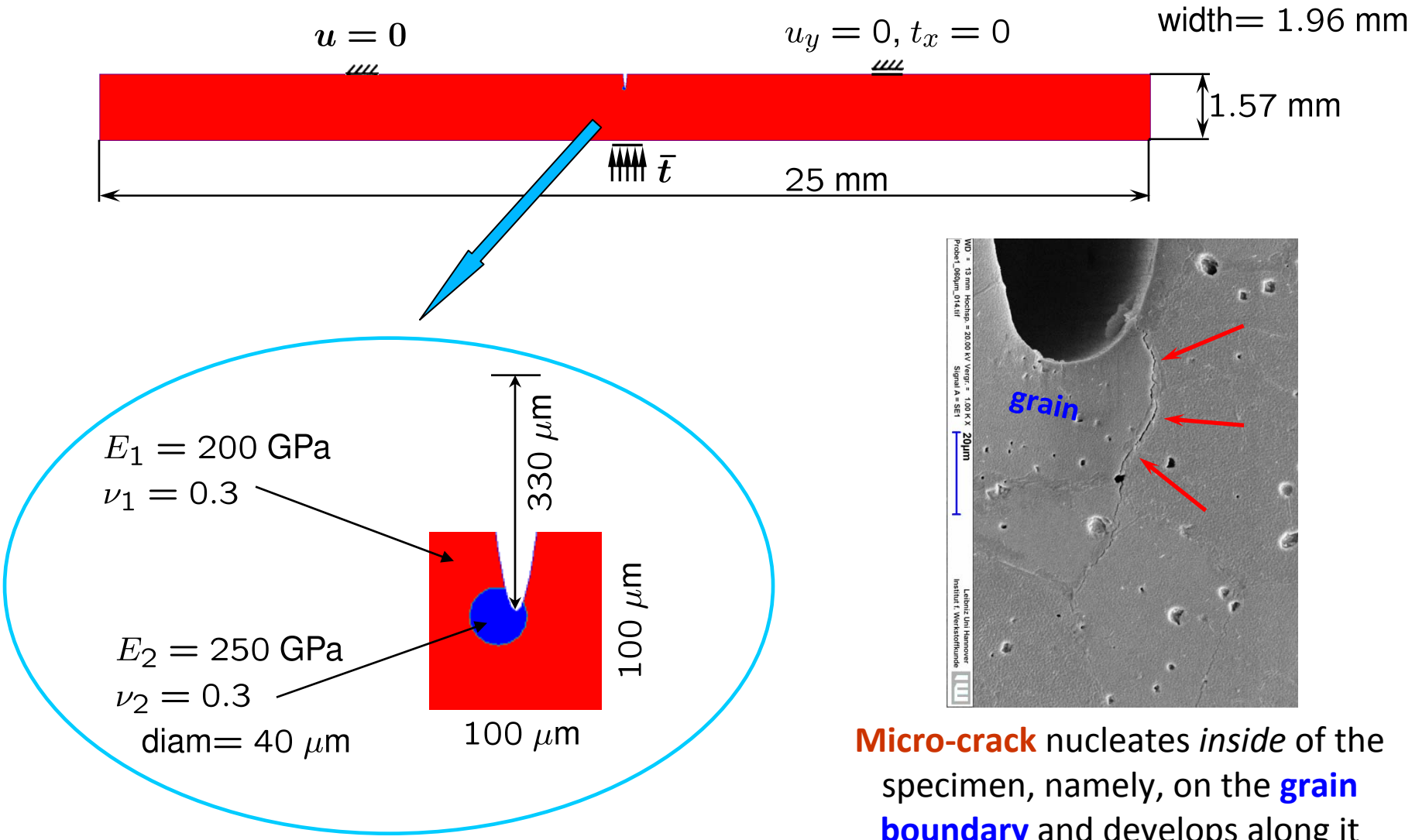


Let's consider only one grain (an inclusion in blue), perfectly bonded with the outer material (in red) and such that $E_1 \neq E_2$

Application III:

Micro-crack initiation in a ceramic specimen (first attempt)

Our setup:



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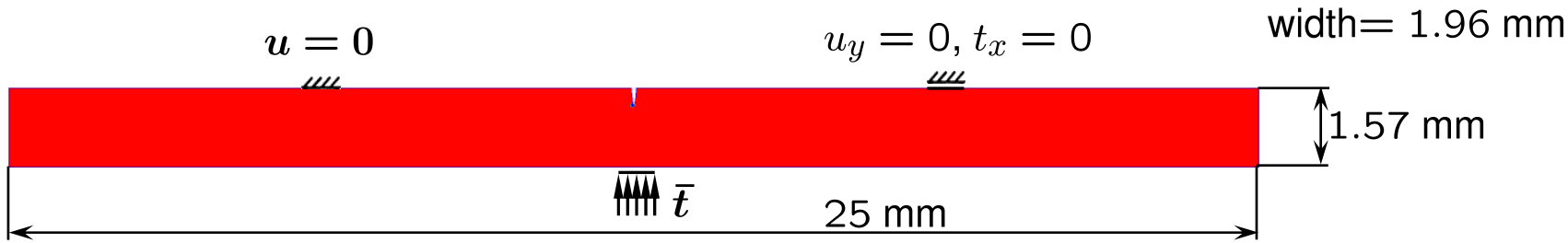
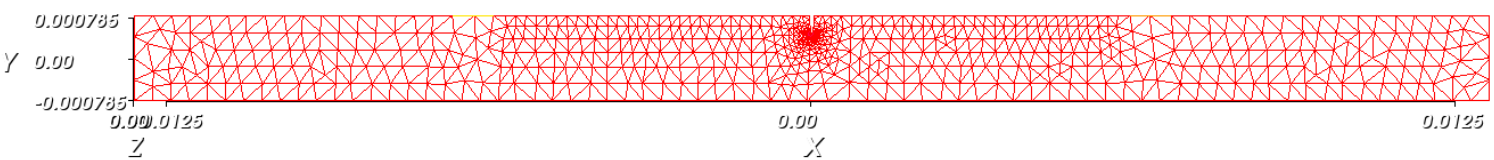
Micro-crack initiation in a ceramic specimen (first attempt)

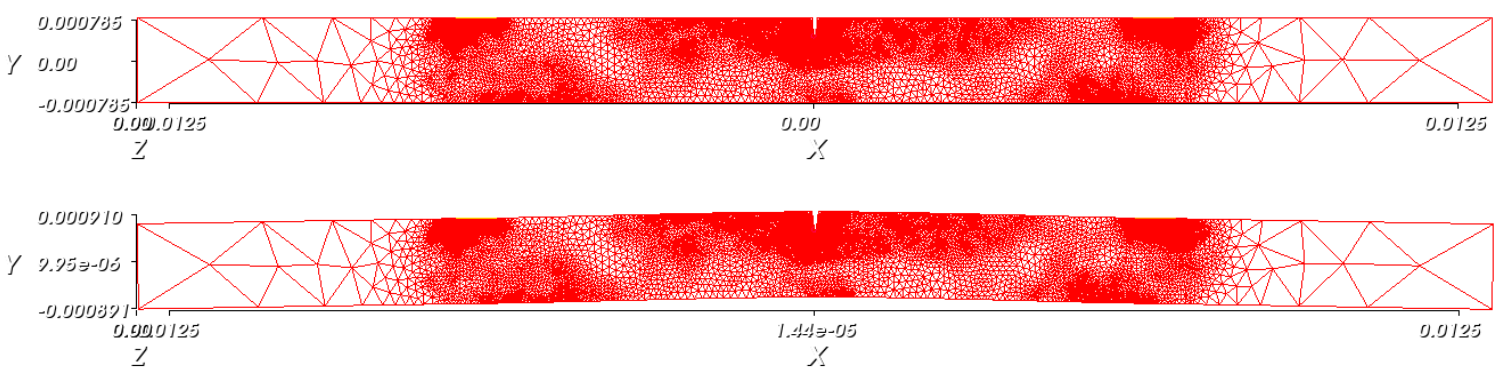
Our setup:



Initial coarse mesh



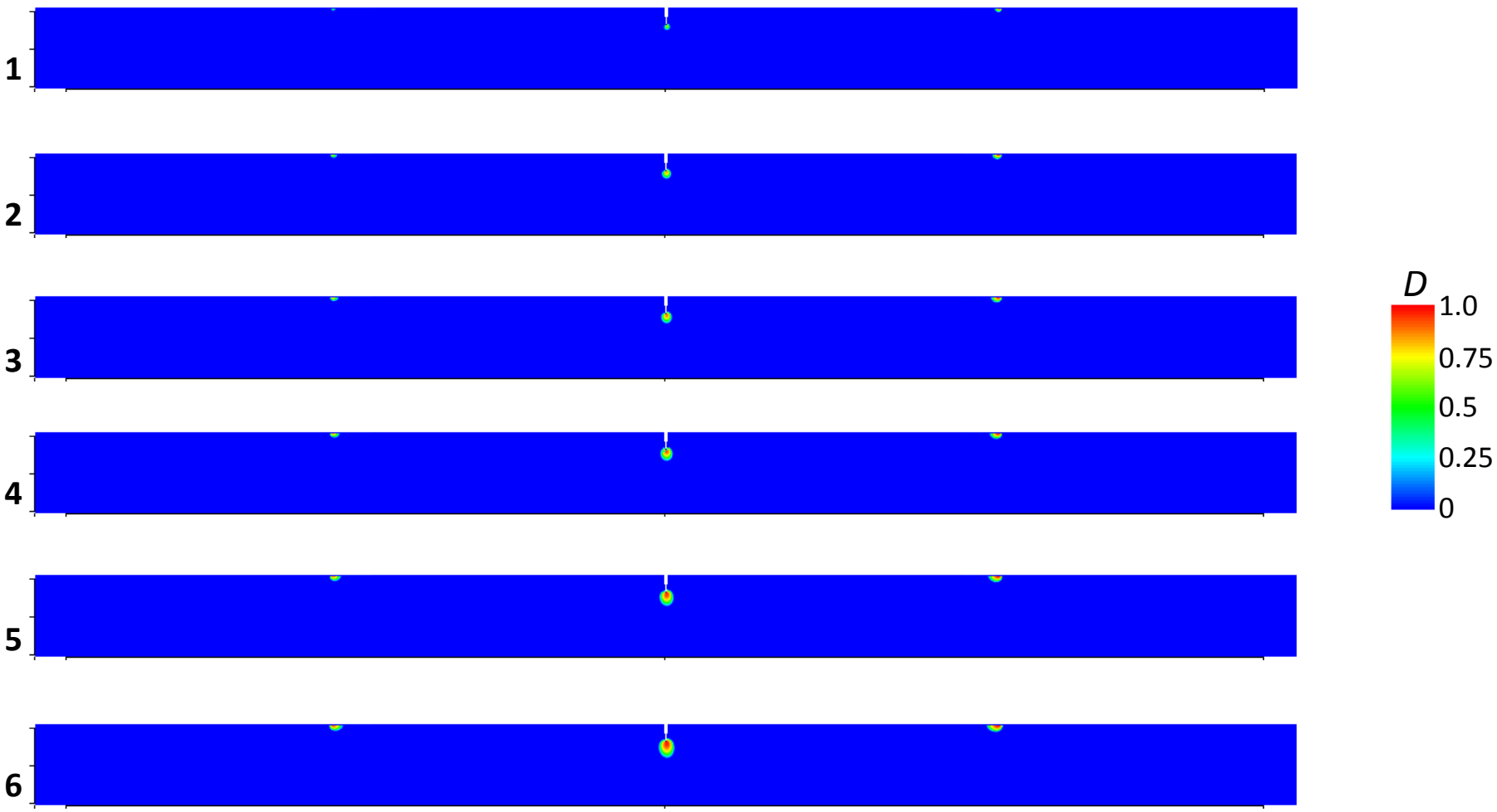
Adapted mesh (before initiation of damage)



Application III:

Micro-crack initiation in a ceramic specimen (first attempt)

Evolution of damage *D* (before the peak load)



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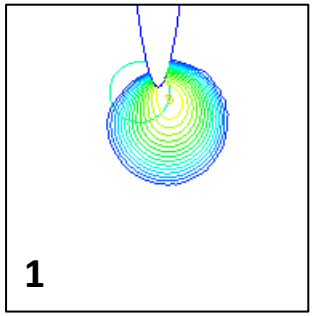
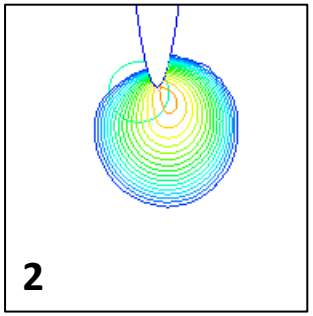
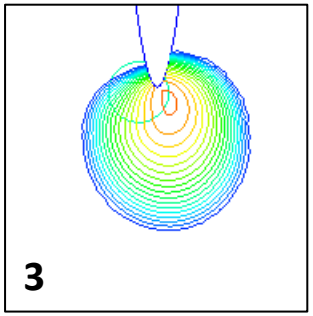
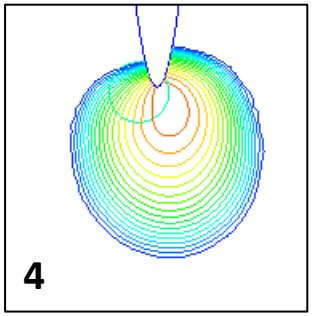
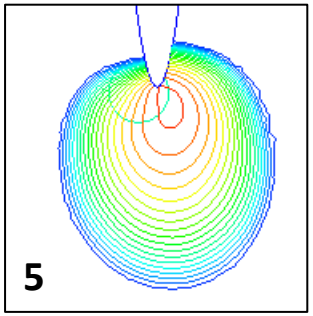
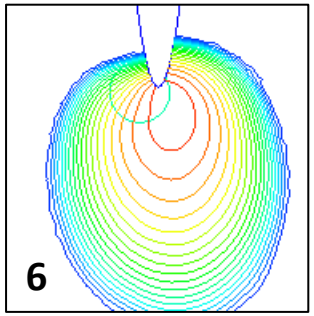


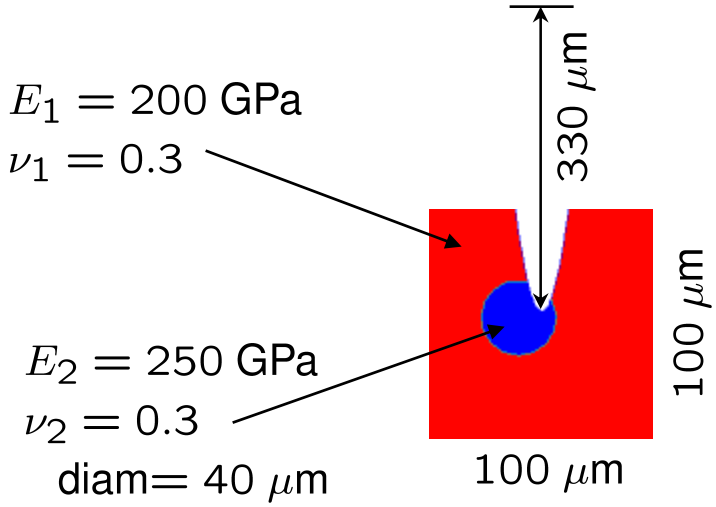








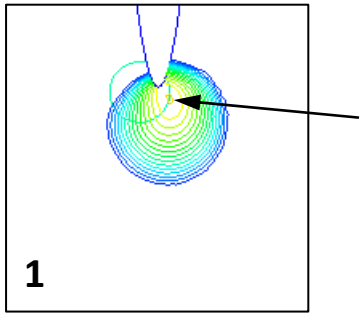




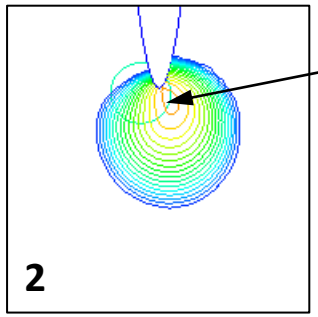
Application III:

Micro-crack initiation in a ceramic specimen (first attempt)

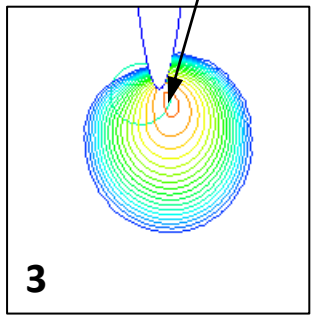
Evolution of damage *D* (before the peak load)



D_{max}

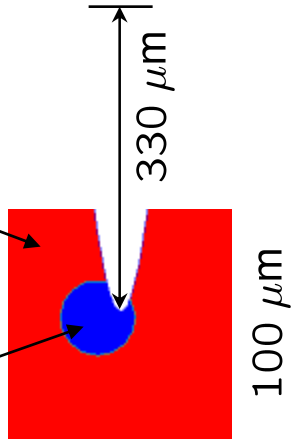


D_{max}

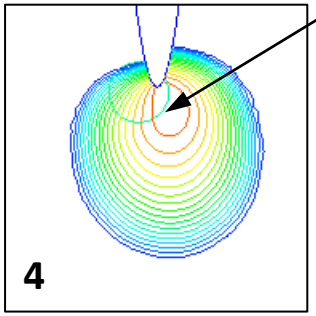


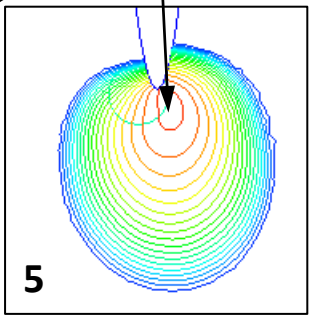
$E_1 = 200$ GPa
$\nu_1 = 0.3$

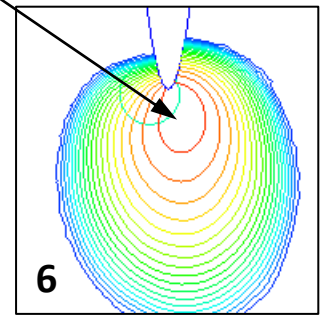
$E_2 = 250$ GPa
$\nu_2 = 0.3$
diam$= 40\ \mu$m



D_{max}





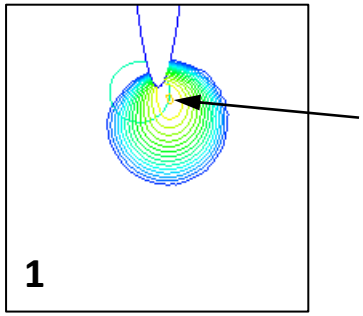


D_{max} is attained on the grain boundary

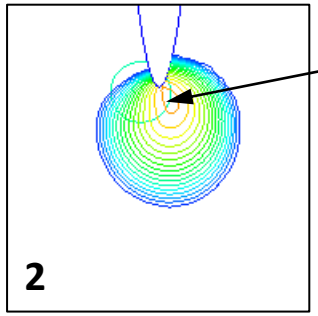
Application III:

Micro-crack initiation in a ceramic specimen (first attempt)

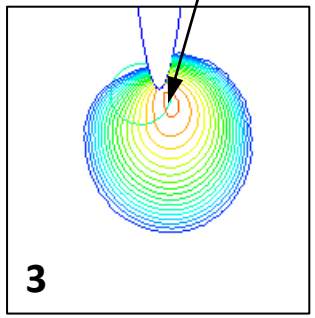
Evolution of damage *D* (before the peak load)



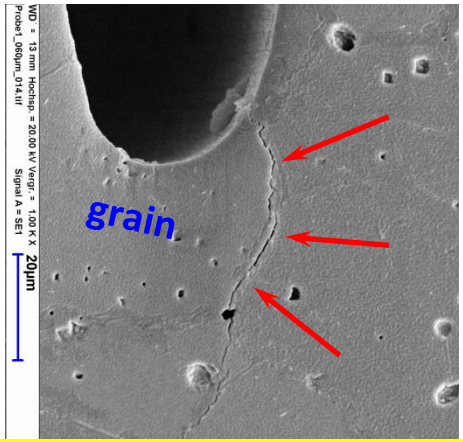
D_{max}



D_{max}

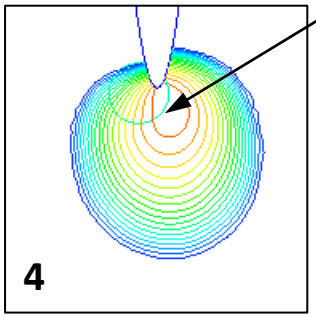


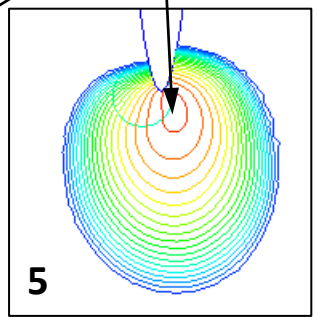
compare

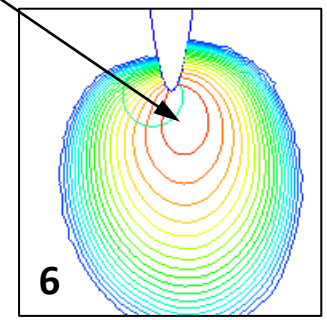


Micro-crack nucleates *inside* of the specimen, namely, on the **grain boundary**

D_{max}







D_{max} is attained on the grain boundary

New Error Estimator: conclusions

$$\|\mathbf{e}\|_{\Omega} \le \frac{\tilde{c}_P \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE}\lambda}} \left(\sum_T \eta_{T,New}^2 \right)^{1/2} =: UB_{New}$$

$$\eta_{T,New} := h_T \left\| \mathbf{f} + \mathrm{div}\boldsymbol{\sigma}(\mathbf{u}^h) \right\|_{\mathbf{L}^2(T)} + \frac{h_T}{|T|^{1/2}} \sum_{\ell=1}^{3} |E_\ell|^{1/2} \left\| \left\langle \boldsymbol{\sigma}(\mathbf{u}^h) \cdot \mathbf{n} \right\rangle_{E_\ell} \right\|_{\mathbf{L}^2(E_\ell)}$$

$$\tilde{c}_P = \frac{4(\sqrt{17}-1)^{1/2}}{(7+\sqrt{17})(3+\sqrt{17})^{1/2}}$$

$$\tilde{c}_K = 2\left(\frac{\pi}{3\pi+2} \right)^{1/2}$$

$$\tilde{c}_{SE} = \frac{2\pi+4}{3\pi+2}$$

- **Explicit** error estimator for 2D (linear) problems
- **Simple** and **cheap**
- *UB* is **guaranteed** (no overestimation)
- *UB* is **accurate** (eff.ind. < 2 => practically acceptable)

- Extension to 3D (linear) problems is seemingly straightforward

[1] T. Gerasimov, E. Stein and P. Wriggers (2012) Fully adaptive FEM/SFM for 2D crack propagation and damage analysis using a new simple constant-free explicit error estimator, *in preparation*.

New Error Estimator: ongoing research

$$\|\mathbf{e}\|_{\Omega} \le \frac{\tilde{c}_P \tilde{c}_K}{\sqrt{2\mu + \tilde{c}_{SE}\lambda}} \left(\sum_T \eta_{T,New}^2 \right)^{1/2} =: UB_{New}$$

$$\eta_{T,New} := h_T \left\| \mathbf{f} + \operatorname{div}\boldsymbol{\sigma}(\mathbf{u}^h) \right\|_{\mathbf{L}^2(T)} + \frac{h_T}{|T|^{1/2}} \sum_{\ell=1}^{3} |E_\ell|^{1/2} \left\| \left\langle \boldsymbol{\sigma}(\mathbf{u}^h) \cdot \mathbf{n} \right\rangle_{E_\ell} \right\|_{\mathbf{L}^2(E_\ell)}$$

$$\tilde{c}_P = \frac{4(\sqrt{17}-1)^{1/2}}{(7+\sqrt{17})(3+\sqrt{17})^{1/2}}$$

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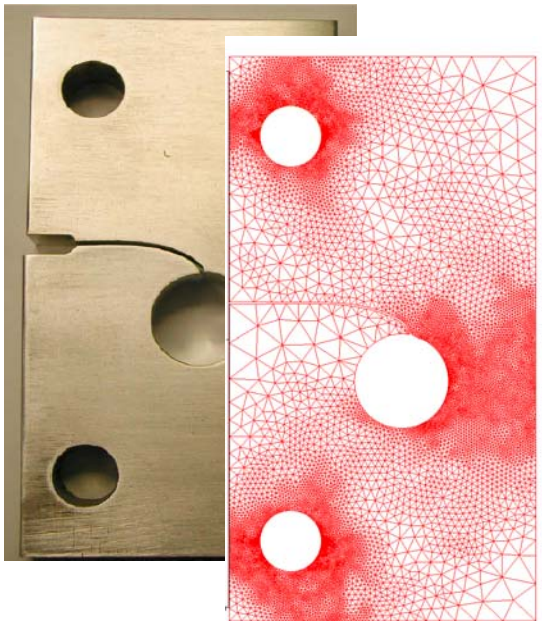
$$\tilde{c}_{SE} = \frac{2\pi+4}{3\pi+2}$$

- Extension to *P*2-,… and *Q*1-,*Q*2-,… based FEMs in 2D
- Extension to 3D
- Goal-oriented EE analysis, $\left| Q(\mathbf{u}) - Q(\mathbf{u}^h) \right| \le \|\mathbf{e}\|_{\Omega} \|\mathbf{e}^*\|_{\Omega}$
- Extension to e.g. XFEM – done in [1], but constant-free (**???**)

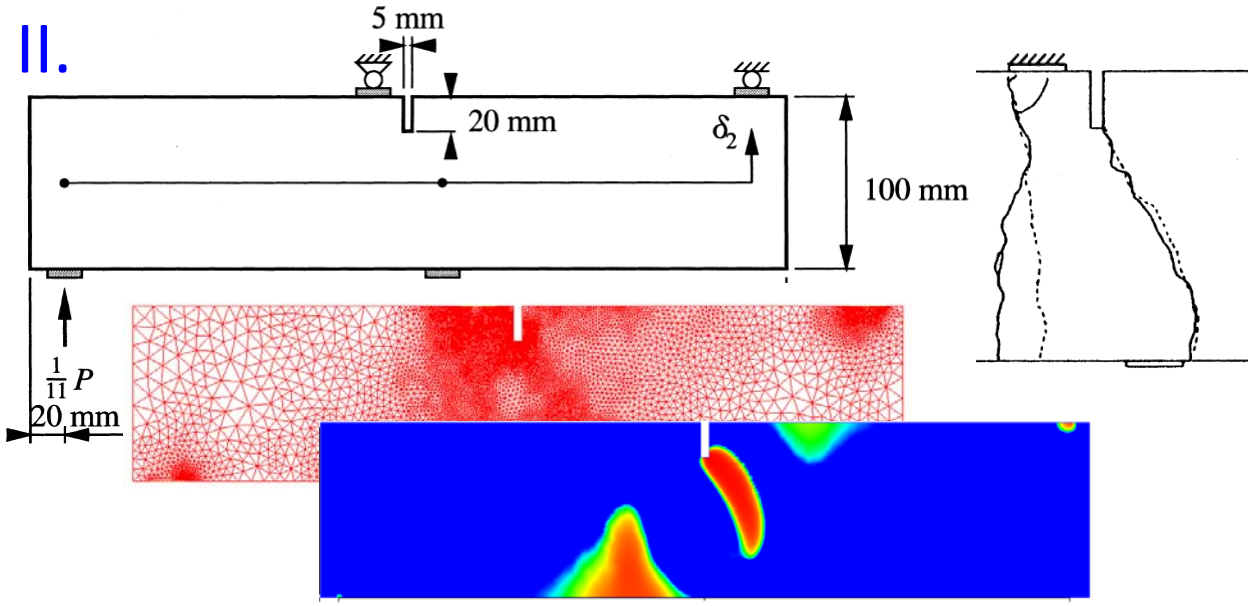
[1] T. Gerasimov, M. Rüter and E. Stein "An explicit residual-type error estimate for Q1-quadrilateral XFEM in LEFM", *IJNME* 2012; **90**:1118-1155.

Applications: conclusions

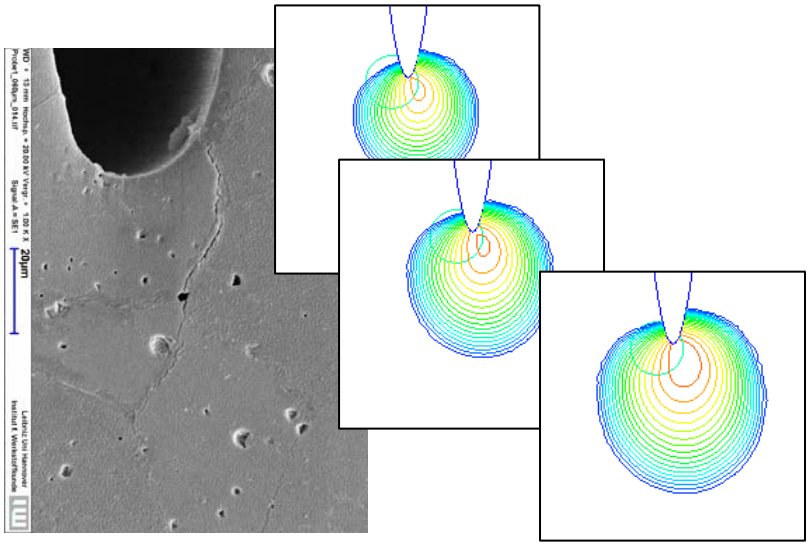
I.



II.



III.



A simple (lower-order) FE technique, equipped with an adequate a posteriori error estimator (to provide efficient, simple and cheap adaptivity), may be rather competitive to the advanced, yet more cumbersome and not-so-easy-to-implement FE techniques like e.g. XFEM/GFEM

Appendices

(planed for the next seminar)

- How does *C* appears?

$$\left\|\mathbf{e}\right\|_{\Omega} \leq \frac{C}{\sqrt{2\mu}}\left(\sum_{T}\eta_{T,Class}^{2}\right)^{1/2} =: UB_{Class}$$

Appendices
(planed for the next seminar)

- How does *C* appears?

$$\|\mathbf{e}\|_{\Omega} \le \frac{C}{\sqrt{2\mu}}\left(\sum_{T}\eta_{T,Class}^{2}\right)^{1/2} =: UB_{Class}$$

- Derivation of

$$\|\mathbf{e}\|_{\Omega} \le \frac{\tilde{c}_P\tilde{c}_K}{\sqrt{2\mu+\tilde{c}_{SE}\lambda}}\left(\sum_{T}\eta_{T,New}^{2}\right)^{1/2} =: UB_{New}$$

$$\eta_{T,New} := h_T\left\|\mathbf{f}+\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}^h)\right\|_{\mathbf{L}^2(T)} + \frac{h_T}{|T|^{1/2}}\sum_{\ell=1}^{3}|E_\ell|^{1/2}\left\|\left\langle\boldsymbol{\sigma}(\mathbf{u}^h)\cdot\mathbf{n}\right\rangle_{E_\ell}\right\|_{\mathbf{L}^2(E_\ell)}$$

$$\tilde{c}_P = \frac{4(\sqrt{17}-1)^{1/2}}{(7+\sqrt{17})(3+\sqrt{17})^{1/2}} \qquad \tilde{c}_K = 2\left(\frac{\pi}{3\pi+2}\right)^{1/2} \qquad \tilde{c}_{SE} = \frac{2\pi+4}{3\pi+2}$$