A primal discontinuous Petrov-Galerkin
Finite Element Method for linear elasticity

Introduction
The focus of this research topic is on the in-
roduction of a novel discretization method,
which is able to overcome the phenomenon
of volume locking. The basis for this is the
discontinuous Petrov-Galerkin (dPG) Finite
Element Method (FEM) proposed recently
by L. Demkowicz and J. Gopalakrishnan
in [1,2] for the transport equation and by
C. Carstensen et al. in [3] for the Poisson
model problem.

Formulation
In this section we provide the derivati-
on of the weak form of the dPG FEM.
We consider a domain Ω with the go-
verning equations for linear elasto-statics:
\[
\begin{align*}
\text{div}(\sigma(\varepsilon)) + \rho b &= 0 \quad \text{in } \Omega \\
\sigma \cdot n &= \tilde{t} \quad \text{on } \Gamma_N \\
u &= \bar{u} \quad \text{on } \Gamma_D \\
\sigma(\varepsilon) &= \mathbf{C} : \varepsilon(u) \quad \text{in } \Omega
\end{align*}
\]
Herein, \(\sigma\) denotes the stress tensor, \(\rho\) the
mass density and \(b\) the body forces in the
domain \(\Omega\). The expression for the static
boundary condition, \(\sigma \cdot n = \tilde{t}\), with the
normal vector \(n\), is defined on the Neumann
boundary \(\Gamma_N\) as the traction vector \(\tilde{t}\). Pre-
scribed displacements on the Dirichlet
boundaries \(\Gamma_D\) are denoted by \(\bar{u}\), and the
linear elastic relation between stresses \(\sigma(\varepsilon)\)
and strains \(\varepsilon(u)\) are achieved by the iso-
tropic fourth order material tensor \(\mathbf{C}\).
The general weak form can now be obtained by
multiplication of the balance of linear mo-
moment with a test function \(\eta\) and integra-
tion by parts on a single triangle \(T\) of the
triangulation \(\mathcal{T}\) and yields
\[
\int_{\mathcal{T}} \sigma(\varepsilon) : \text{grad}(\eta) \cdot dA - \int_{\partial T} \sigma \cdot n_T \cdot \eta \cdot ds = \int_{\partial T} \rho b \cdot \eta \cdot dA,
\]
Besides the classical interpolation of element
displacements \(\mathbf{u}\), the occurring boundary
integral \(\partial T\) with the tractions \((\sigma \cdot n_T)\) at the surface of adjoined elements
is also evaluated. In this context an addi-
tional variable \(t_T\) for the tractions is in-
troduced resulting in a mixed formulation.
The weak form for the primal discontinuous
Petrov-Galerkin FEM can now be obtained by
summing up over all triangles \(T\) of the
conditions the system can be solved directly
for the unknowns \(\bar{u}\) and \(\tilde{t}\).

Results
In an example, the performance of the pro-
posed formulation is tested for the well-
known Cook's Membrane problem.

Fig. 1: Convergence of vertical
displacement of the top right point.
The convergence of the vertical displace-
ment during mesh refinement for different
classical Galerkin formulations is compa-
red to the proposed T1-dPG formulation
(Fig. 1). The T1-dPG formulation has a bet-
ter convergence than elements with linear
interpolations (T1 & Q1), but is still insuf-
ficient compared to higher-order elements
(T2 & Q2) in the nearly incompressible case
where \(\nu = 0.499\).

Fig. 2: Displacement solution for
Cook's membrane.

Outlook
In a next step, investigations on the ex-
tension and implementation of a nonlinear
primal discontinuous Petrov-Galerkin Finite
Element Method (e.g., St. Venant or Neo-
Hookean material) are conducted.

References
[1] Demkowicz, L. and Gopalakrishnan, J.: A
class of discontinuous Petrov-Galerkin methods.
Part I: The transport equation
class of discontinuous Petrov-Galerkin methods.
II. Optimal test functions
for an elliptic PDE