Direct nodal imposition of surface loads using the divergence theorem

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Abstract For the accurate imposition of surface loads using the Finite Element Method normally the load is discretized at the surface facets. Therefore appropriate surface shape functions are needed. If the surface contains kinks, which occurs often in contact cases, the imposition is more complicated. In order to simplify the imposition of surface loads, an alternative approach is presented. This formulation is purely based on volume contributions of the discretized elements of the body. The surface nodes are automatically identified and the linearization is straightforward. No specific surface information is necessary. The imposition is simply based on the application of the divergence theorem. The only prerequisite is the fulfillment of the integration constraint, a necessary requirement for Galerkin solution schemes. With this, boundary nodes are directly identified by possessing a non zero normal vector whereas for inner nodes this vector is identically zero. Moreover, the normal vectors at the boundary nodes contain all the information of the surface and correspond to resultant nodal normal vectors. This is especially advantageous in the case of surfaces that contain kinks. It also simplifies the search algorithm in computational contact mechanics to find the closest distance to other bodies. This approach also works for any kind of shape functions. The nodal force vectors are always determined accurately. The advantages and the simple handling of this approach are demonstrated by means of several examples including follower loads and contact cases. Additionally, the influence of the isoparametric concept on the integration constraint is investigated by evaluating the behavior of different shape functions on an irregular grid. Although this new approach is only demonstrated within the context of the Finite Element Method, due to its generic derivation it can be applied to any Galerkin solution scheme which fulfills the integration constraint.

Keywords Finite Element Method · Integration constraint · Surface loads · Divergence theorem

1 Introduction

In structural analysis engineers have to deal with different kind of loadings. These can be dead loads or loads due to a gas or a fluid on a structure or on a fluid. The last cases contain for instance the filling of structures with liquids, the inflation of air supported structures or the additive manufacturing of metals.
where surface tension, Marangoni convection or recoil pressure dominate the fusion behavior of metallic particles for instance. In all of these mentioned examples, the loads are described as follower loads. These are defined in the current configuration and depend on the deformation. A detailed discussion on these loadings can be found for instance in Schweizerhof and Ramm [1984] and theoretical examinations on the conservative or the non-conservative character of this type of loading in Sewell [1967] or Simo et al. [1991].

Using weak form based solution schemes, like the Finite Element Method (FEM), to solve the continuum mechanical differential equation surface loads are normally transformed into nodal force vectors. Different procedures are possible, but mostly the force vectors result from an integration of the surface load over the adjacent facets of the corresponding elements (see Zienkiewicz and Taylor [1989] for instance). For follower loads the force vector has to be updated at each load step. Due to the discretization scheme mentioned above this leads to a complex linearization. Details for the non linear case are shown in Schweizerhof and Ramm [1984] or Wriggers [2008]. Examples in the context of the Finite Element Method can be found in Bonet et al. [2000], Rumpel and Schweizerhof [2004] or Rumpel et al. [2005], for instance. The imposition of surface tension, Marangoni convection and recoil pressure using an ALE approach is demonstrated in King et al. [2015].

If the discretized surface contains kinks which often occurs in contact cases, the situation is even more complicated. Using the Mortar method in computational contact mechanics a resultant normal vector at each node is mostly computed based on surface data. However, this leads to additional efforts, see for instance Yang et al. [2005] or Weißenfels and Wriggers [2015]. Alternatively, a smooth surface can be constructed by using either Hermite, Bernstein (Wriggers et al. [2001]) or NURBS interpolation functions (Stadler et al. [2003]). Using such an approach a normal vector perpendicular to the surface can always be defined.

However, an alternative approach is also possible which simplifies the imposition of surface loads and contact computations even for surfaces which contain kinks. No surface information is needed. The normal and the force vectors are completely determined based on discretized volume element quantities only. Whereas only the nodes at the boundary have a normal vector which is non zero in length. Hence, the boundary can be directly defined with this approach. Additionally, the resultant nodal normal vectors are automatically determined. The scheme presented here is just based on the divergence theorem and on a condition on test functions defined in Krongauz and Belytschko [1997] which is also named integration constraint in Chen et al. [2001]. Due to its generic derivation the presented approach can be applied to any Galerkin solution scheme which fulfill the integration constraint.

The derivation of this alternative approach is sketched in section 3. Before, the nodal discretization of the balance of momentum using the Finite Element Method is introduced in section 2. Several examples in section 4 show the plurality of applications where this approach is advantageous. This includes a contact case, the compression of a structure with a gas filled hole and an cuboid container which is loaded by a hydrostatic fluid.

2 Finite Element discretization

The balance of momentum is solved computationally using the Finite Element Method. The weak form and the concept of nodal discretization are described in section 2.1. In this work only an elastic material behavior is considered and the corresponding stress strain relationship is given in section 2.2. To demonstrate the applicability of the proposed approach to impose surface loads different shape functions are used which is presented in section 2.3. The integration over the volume (section 2.4) is evaluated
2.1 Weak form and nodal discretization

In order to concentrate on the imposition of surface loads within the continuum mechanical differential equation inertia terms and the self weight of the structure are neglected. Hence, the weak form balances the virtual work of the traction vector $\mathbf{t}$ at the outer surface with the virtual work within the body written here with respect to the current configuration

$$G(\eta, u) = \int_{\Omega_t} \text{grad} \eta \cdot \sigma(u) \, dv - \int_{\partial B_t^e} \eta \cdot \mathbf{t} \, da = 0. \quad (1)$$

The Cauchy stress tensor $\sigma$ represents the material behavior and is determined in section 2.2. Using the Finite Element Method to solve the above differential equation the domain is subdivided into non-overlapping elements and the integration is evaluated numerically. Hence, the continuum under investigation is subdivided into nodes $I$ and integration points $p$, see also figure 1. In the discretization process the primary variable (or trial function) $u$ and the test function $\eta$ at the current load step $n + 1$ are approximated within the element by means of shape functions $N_I$.

$$\eta_{p,n+1} = \sum_{i=1}^{N} N_{I,n+1} \eta_{I,n+1}, \quad u_{p,n+1} = \sum_{i=1}^{N} N_{I,n+1} u_{I,n+1} \quad (2)$$

where $N$ corresponds to the number of nodes in each element. In order to derive the alternative scheme to impose surface loads a nodal wise assembly procedure is preferred

$$R(u_{n+1}) = \mathbf{A} \sum_{I=1}^{n_{nod}} \sum_{p=1}^{N_{I,n+1}} \mathbf{B}_{I,p,n+1} \sigma_{p,n+1} v_{p,n+1} - P_{n+1} = 0. \quad (3)$$

The residual vector $R$ contains the discretized form of the inner virtual work and the contribution from the surface loads which are summarized in the vector $P$. The influence domain $N_{I,n+1}$ of node $I$ contains all integration points located in the adjacent elements, see also figure 1. Furthermore, $v_{p,n+1}$ indicates the current volume contribution of the integration point $p$, $\mathbf{A}$ the assembly operator and $n_{nod}$ is the total number of nodes. In this work, the external force vector $P$ is derived only from volume contributions together with the divergence theorem as shown in detail in section 3. The $\mathbf{B}$-matrix in the above equation contains the derivatives of the shape functions. Detailed information on the discretization process and the resulting equations can be found in standard textbooks on the FEM, like Zienkiewicz and Taylor [1989] or Wriggers [2008].

As a consequence of the nodal assembly procedure the discretized balance of momentum can be set up for each node individually. The inner nodal force vectors have to be zero and the boundary force vectors have to be equal to the external force at that node

$$f_{I,n+1} = \sum_{p=1}^{N_{I,n+1}} \mathbf{B}_{I,p,n+1} \sigma_{p,n+1} v_{p,n+1} = \begin{cases} 0 & \text{inner node} \\ f_{I,n+1}^{ext} & \text{boundary node} \end{cases}. \quad (4)$$
2.2 Constitutive equation

Within this work only elastic material behavior is investigated. A formulation describing a nonlinear stress strain relation using two material parameters \( \lambda, \mu \), which directly depend on the Young’s modulus \( E \) and the Poisson ratio \( \nu \), is the Neo-Hookean model

\[
\sigma_{p,n+1} = 2 b_{p,n+1} \frac{\partial W(b_{p,n+1})}{\partial b_{p,n+1}} = \frac{\lambda}{2 J_{p,n+1}} (J_{p,n+1}^2 - 1) + \frac{\mu}{J_{p,n+1}} (b_{p,n+1} - 1). \tag{5}
\]

The left Cauchy Green tensor \( b_{p,n+1} \) and the Jacobian \( J_{p,n+1} \) can be directly computed from the deformation gradient \( F_{p,n+1} \) via

\[
b_{p,n+1} = F_{p,n+1} F_{T_{p,n+1}}^{-1} \quad \text{and} \quad J_{p,n+1} = \det F_{p,n+1}.
\]

In this work, the deformation gradient and the shape function derivatives with respect to the current coordinates are computed using a Lagrangian formulation. Hence, the current deformation gradient and the shape function derivatives compute only from the derivatives with respect to the initial coordinate \( X_p \)

\[
F_{p,n+1} = 1 + \sum_{I=1}^{N} u_{I,n+1} \otimes \frac{\partial N_I(X_p)}{\partial X_p}, \quad \frac{\partial N_I(X_p)}{\partial X_p} = \frac{\partial N_I(X_p)}{\partial X_{p,n+1}} F_{p,n+1}^{-1} F_{T_{p,n+1}}^{-1}.
\tag{6}
\]

Due to the nonlinear material model equation (3) is also nonlinear with respect to the primal variable. The Newton Raphson algorithm is applied to determine the resulting displacements at every load step. Detailed information about this scheme can be found for instance in Wriggers [2008]. The current displacements of each node result then from the update using the increment of the displacements \( \triangle u \)

\[
u_{I,n+1} = u_{I,n} + \triangle u_{I,n+1}.
\tag{7}
\]

2.3 Shape functions

In order to demonstrate that this approach automatically computes the correct force vectors at each boundary node four different shape functions are investigated which are commonly used in Galerkin solution schemes. The first one is the standard polynomial basis function which is normally applied within the Finite Element method. Additionally, the Local Maximum Entropy (LME) basis functions (Arroyo and Ortiz [2006]), the Moving Least Square (MLS) functions (Lancaster et al. [1981]) and the 1-dimensional Wendland \( C^2 \) basis functions (Wendland [1995]) are examined within the context of the
Finite Element Method. These functions are commonly used in the Optimal Transportation Meshfree (OTM) method (Li et al. [2010], Weißenfels and Wriggers [2018]), the Element Free Galerkin method (Belytschko et al. [1994]) and the Smoothed Particle Hydrodynamics (SPH) like in Monaghan and Kajtar [2009], respectively.

Following Strang and Fix [1973] zero and linear completeness in the trial function are sufficient conditions for convergence in a Galerkin method. The polynomial basis function possesses all these requirements and is presented in detail in many books dealing with the Finite Element method, like Hughes [1987], or Wriggers [2008]. Hence, a study of these is omitted in this work.

In order to ease the notation the subscript \( n+1 \) is neglected in the following sections. If an indication of the load step is necessary, it will be noted at the appropriate place.

### 2.3.1 Local Maximum Entropy shape functions

The LME shape functions are special exponential radial basis functions where the zero order completeness condition is simply fulfilled by a normalization

\[
N_I(\mathbf{X}_p) = \frac{Z_I(\mathbf{X}_p)}{Z} \quad Z_I(\mathbf{X}_p) = e^{-\beta \|\mathbf{X}_p - \mathbf{X}_I\|} + \lambda_p [\mathbf{X}_p - \mathbf{X}_I], \quad Z = \sum_{I=1}^{N} Z_I(\mathbf{X}_p).
\]  

(8)

The unknown Lagrange multiplier \( \lambda \) is computed by enforcing the first order completeness constraint to hold

\[
r(\mathbf{X}_p, \lambda_p) = \sum_{I=1}^{N} N_I(\mathbf{X}_p) [\mathbf{X}_p - \mathbf{X}_I] = 0.
\]  

(9)

The LME shape function also possesses a weak Kronecker-\( \delta \) property, see figure 2. This allows the direct imposition of Dirichlet boundary conditions, if the boundary is convex.

![Figure 2: Distribution of the Local Maximum Entropy (LME) shape function and derivatives of the lower left node.](a) Shape function (b) Derivative x-direction (c) Derivative y-direction)

### 2.3.2 Moving least square shape functions

In order to use polynomial basis functions even for arbitrary number of nodes moving least square functions are preferred in this cases. The shape functions are computed by minimizing the error of the difference between the nodal value and its approximation. Since the shape function computation rapidly
deteriorates, if the number of nodes is much larger than the order of the polynomial (see for instance Onate et al. [1996]), the least square approximation is enhanced by a scalar weighting functions $w$

$$N_I (X_p) = w_I (X_p) p(X_p) \cdot (M(X_p)^{-1} P_I), \quad M(X_p) = \sum_{I=1}^{N} w_I (X_p) P_I \otimes P_I. \quad (10)$$

The length of the vectors $p$ and $P$ depends on the order of the polynomial basis function. The vector $p$ describes the polynomial at each integration point and $P$ the polynomial at the corresponding node $I$. The function $w_I$, sometimes also called window function, is mostly either a Gaussian or a polynomial function. Here a cubic spline function is applied

$$w_I(X_p) = \begin{cases} 
\frac{2}{3} - 4r^2 + 4r^3 & 0 \leq r \leq \frac{1}{2} \\
\frac{4}{7} - 4r + 4r^2 - \frac{4}{7}r^3 & \frac{1}{2} < r \leq 1 \\
0 & r > 1
\end{cases}, \quad r = \frac{\|X_p - X_I\|}{d_I} \quad (11)$$

In this work, the parameter $d_I$ corresponds to the maximal distance between an integration point and a node within the element multiplied by the factor 1.05. The MLS shape functions are zero and first order complete, but do not possess the Kronecker-$\delta$ property, which can be seen figure 3.

![Figure 3: Distribution of the Moving Least Square (MLS) shape function using a linear polynomial basis function and derivatives of the lower left node.](image)

(a) Shape function  \hspace{5em} (b) Derivative x-direction  \hspace{5em} (c) Derivative y-direction

2.3.3 \textit{Wendland C$^2$}

Another way to approximate the field in an element is to use any kind of radial basis functions together with a correction. Here, the 1-dimensional Wendland C$^2$ approximation function is used exemplarily. The zero order completeness requirement can be obtained by normalizing the shape functions

$$N_I (x_p) = \frac{w_I (X_p)}{W}, \quad w_I (X_p) = (1 + \frac{3}{2}r) * (2 - r)^3, \quad W = \sum_{I=1}^{N} w_I (X_p). \quad (12)$$

with $r$ being the same as in equation (11). Instead of the first order completeness condition its derivatives are forced to hold

$$\text{first order} \sum_{I=1}^{N} N_I (X_p) X_I = X_p, \quad \text{derivative first order} \sum_{I=1}^{N} X_I \otimes \frac{\partial N_I}{\partial X_p} = 1. \quad (13)$$
This can be easily achieved by correcting the shape function derivatives with the inverse from the first order completeness condition

\[
\frac{\partial N_I}{\partial x_p} = \frac{\partial N_I}{\partial x_p} L^{-1}, \quad L = \sum_{I=1}^{N} X_I \otimes \frac{\partial N_I}{\partial x_p}.
\] (14)

This scheme is mostly applied using the SPH solution scheme, see Randles and Libersky [1996]. However,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Distribution of the corrected 1-dimensional Wendland $C^2$ shape function and derivative of the lower left node.}
\end{figure}

the 1-dimensional Wendland $C^2$ function also does not bear the Kronecker-$\delta$ property, as shown in figure 4.

2.4 Numerical integration

For the computation of the algebraic set of equations the integrals have to be evaluated. In the FEM the isoparametric concept (Irons and Zienkiewicz [1968]) is normally applied where the shape functions and their derivatives are computed on a regular local coordinate system. The derivatives with respect to the global coordinates are obtained by a transformation using the Jacobi Matrix $J$. In the standard FEM the integrals are mostly numerically evaluated using the Gauss quadrature rule. In this case, the volume contribution of each integration point is computed from the determinant of the Jacobi matrix multiplied with the weighting $W_p$ of the Gauss quadrature rule. Using the Lagrangian description only the initial volume is computed. The current volume can be obtained by a push-forward using the determinant of the deformation gradient $v_0 = \det J_0 W_p, \quad v_{p,n+1} = \det F_{p,n+1} v_0.$ (15)

For polynomial basis functions this leads to an accurate integration over the domain, if the shape of the finite element is not too deformed. For the other shape functions an accurate integration is hardly to achieve. Nevertheless, for these cases the concept of a Gaussian integration leads to satisfactory results. Only the number of integration points has to be increased for an accurate solution. Additionally, the LME, MLS and Wendland $C^2$ shape functions can also be computed using the isoparametric concept as investigated in section 3. Details of the isoparametric concept and the numerical integration can be found in standard FEM textbooks, like Hughes [1987] or Wriggers [2008]. A detailed list of weightings $W_p$ for different number of integration points is given in Dhatt and Touzot [1984].
3 Imposition of traction boundary conditions

Beside the completeness condition another criterion has to be fulfilled by Galerkin solution schemes. This condition can be directly derived from a nodal wise assembly procedure. If the stress is constant in the influence domain, equation (4) can be reformulated

\[ f_{I, i} = \sum_{p=1}^{N_i^{in}} N_{I, j} p \sigma_{p, ij} v_p = \left( \sum_{p=1}^{N_i^{in}} N_{I, j} p v_p \right) \sigma_{p, ij} = 0, \quad i, j = 1, 2, 3. \]  

(16)

Since the stress can be arbitrary, for inner nodes the condition

\[ \sum_{p=1}^{N_i^{in}} N_{I, j} p v_p = 0 \]  

(17)

has to hold. This requirement was formulated in Krongauz and Belytschko [1997], named integration constraint in Chen et al. [2001] and ensures that a constant stress in the influence domain leads to a zero force vector at inner nodes. Since the above equation can also be interpreted as the discretized divergence operator, this requirement can be further transformed. Following the divergence theorem, which can be found for instance in Holzapfel [2000], equation (17) can also be interpreted as the sum over all nodal normal vectors \( n_f^e \) of each adjacent element

\[ \sum_{p=1}^{N_i^{in}} N_{I, j} p v_p = \sum_{e=1}^{N_i^{in}} n_f^e = 0. \]  

(18)

The nodal normal vector of each element computes by means of the sum over all Gauss points of the corresponding element \( n_{g_p}^e \)

\[ n_f^e = \sum_{p=1}^{n_{g_p}} N_{I, j} p v_p. \]  

(19)

If the integration constraint is fulfilled, the sum over all individual nodal normal vectors is zero for inner nodes and non zero for outer nodes

\[ \sum_{e=1}^{N_i^{in}} n_f^e = \begin{cases} 0 & \text{inner nodes} \\ n_f^e & \text{boundary nodes} \end{cases}. \]  

(20)

This condition is also sketched graphically in figure 5. Hence, inner nodes and nodes at the boundary can be directly identified by means of volume element information only.

Additionally, the nodal normal vector at the boundary corresponds to the resultant nodal normal vector which directs outwards at the surface. This vector also includes information about the area which belongs to that node \( a_f^e = \| n_f^e \| \)

\[ \sum_{p=1}^{N_i^{in}} N_{I, j} p v_p = \sum_{e=1}^{N_i^{in}} n_f^e a_f^e, \quad n_f^e = \frac{n_f^e}{\| n_f^e \|}. \]  

(21)

A multiplication of the nodal normal vector with an uniform surface load \( q \) results in a force vector which acts at this node

\[ f_{I, i}^{ext} := q n_f^e = q \sum_{p=1}^{N_i^{in}} N_{I, j} p v_p, \quad n_f = \sum_{p=1}^{N_i^{in}} N_{I, j} p v_p. \]  

(22)
Using this equation for the imposition of uniform surface loads no surface parametrization is needed. The nodal force vector computes directly from the weak form of the differential equation. This way of imposing surface loads corresponds to a nodal integration over the surface area.

In case of follower loads, the current normal vector has to be used. Since the equations are formulated with respect to a Lagrangian description, the current normal vector results from a push-forward from the initial to the deformed configuration using the deformation gradient of equation (6)

\[ \mathbf{n}_I = \sum_{p=1}^{N_{I,nf}} \frac{N_I(X_p)}{\partial X_p} \mathbf{F}_p^{-1} \det \mathbf{F}_p \mathbf{v}_0 \]  

(23)

The linearization of the current normal vector is only based on the linearization of the inverse of the deformation gradient and the determinant of \( \mathbf{F} \) and leads in its final form to

\[ \Delta_u \mathbf{n}_I = \sum_{p=1}^{N_{I,nf}} \left[ \frac{\partial N_I}{\partial x_p} \otimes \left( \sum_{j=1}^{N} \frac{\partial N_j}{\partial x_p} \right) \right] \mathbf{v}_p \Delta_u ^{n} \mathbf{u}_J \]

(24)

### 3.1 Influence of shape functions

The accurate imposition of surface loads for different shape functions is studied by a simple 2-dimensional example. A rectangular elastic block of thickness 1 mm is compressed by a dead load of 20 N/mm². The material behavior is assumed to behave nonlinear elastic and the parameters in equation (5) are chosen as \( E = 7\ \text{N/mm}^2 \), and \( \nu = 0.499 \). The geometry, the dimensions and the boundary conditions can be found in figure 6. The block is discretised using 9-node elements and the domain is subdivided into elements possessing an equal size. This leads to 4 elements in horizontal and 10 elements in vertical direction. In order to demonstrate the automatic adjustment of the proposed approach the node at the vertex A and the node on the edge B of the second 9-node element in horizontal direction at the upper
Figure 6: (Left) Geometry, dimensions and boundary conditions of the block. (Right) Boundary nodes of a single 9-node element which is investigated.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Q2</td>
<td>13.333</td>
<td>6.667</td>
</tr>
<tr>
<td>LME ($\gamma = 2.0$)</td>
<td>5.368</td>
<td>14.632</td>
</tr>
<tr>
<td>MLS</td>
<td>14.441</td>
<td>5.559</td>
</tr>
<tr>
<td>Wendland $C^2$</td>
<td>7.141</td>
<td>12.859</td>
</tr>
</tbody>
</table>

Table 1: Table showing the corresponding vertical nodal forces in dependency of the shape functions.

surface is investigated, see also figure 6. In this case only the upper surface is defined as the loaded surface and the normal vectors are projected onto the direction of the dead loading. Table 1 shows that different shape functions within the element lead to different magnitudes of the force vectors at the corresponding boundary nodes. The magnitude of the force vector at node A and node B are different for the quadratic polynomial (Q2), the Local Maximum Entropy (LME) with the parameter taken as $\gamma = 2.0$, the Moving Least Square (MLS) and the Wendland $C^2$ shape functions. However, all cases result in the same correct homogeneous stress state.

3.2 Influence of isoparametric concept

In the second test the block of the previous example is subdivided into elements with different sizes, as can be seen in figure 7. Since the MLS and the Wendland $C^2$ shape functions do not bear the Kronecker-$\delta$ property, the Dirichlet boundary conditions can not be imposed directly. In this case these constraints are normally weakly imposed using mostly either the Lagrange multiplier or the penalty method, see Belytschko et al. [1994] and Zhu and Atluri [1998]. The concept of transforming Dirichlet into Neumann boundary conditions is similar to imposing surface loads. Hence, this new approach can also be used in this case. Applying the penalty method the force vector at each boundary node results simply from the normal vector

$$f_{I}^{\text{Dirichlet}} = c_N (\mathbf{u}_I - \bar{\mathbf{u}}_I) \cdot \frac{\mathbf{n}_I}{\|\mathbf{n}_I\|} \mathbf{n}_I.$$  
(25)
with $\mathbf{u}_I$ being the imposed Dirichlet boundary condition. The penalty parameter $c_N$ is chosen as $10^6 \text{n/mm}^3$ to allow only small deviations.

Nevertheless, if the size of the elements differ, LME, MLS and Wendland $C^2$ shape functions, defined on global coordinates, lead to inaccurate results. At the inner nodes the normal vectors are not equal to zero anymore, see also figure 7. Hence, the integration constraint is violated which causes the inaccurate results. However, if the isoparametric concept is used for the LME, MLS and Wendland $C^2$ shape functions as well, the normal nodal vectors at the inner nodes are identically zero (see figure 7) and the correct solution is obtained. This shows the advantage of the isoparametric concept where the shape functions are computed on a regular element with unit length. The derivative with respect to the global coordinates result from a push-forward from the reference to the initial configuration by means of the Jacobian tensor $\mathbf{J}$

$$\frac{\partial N_I}{\partial \mathbf{X}} = \frac{\partial N_I}{\partial \xi} \mathbf{J}^{-1}, \quad \mathbf{J} = \sum_{I=1}^{N} X_I \otimes \frac{\partial N_I}{\partial \xi}. \tag{26}$$

4 Further test cases

Different scenarios are investigated in the following sections. The first one (section 4.1) shows the advantage of this approach in contact cases. The second one (section 4.2) investigates the behavior of a gas filled hole within a structure under deformations. The special feature within this example is that the location of the hole is not known before and only determined based on the resultant normal vectors. The last example (4.3) shows the influence of the nodal integration approach. A cuboid container is filled with water resulting in different hydrostatic pressures at the nodes depending on the fluid height. To avoid additional computational efforts before the simulation the nodes at the surfaces are identified where the load is or can be applied. Afterwards, only at these nodes the resultant normal vector is updated if necessary.
4.1 Forming of a block

In the first test example a contact case is computed showing the advantage of the integration constraint in terms of contact search. Therefore, a block is compressed by a wedge. The geometry, the dimensions and the boundary conditions can be found in figure 8. The block is discretized using 8-node hexahedral elements. Again, the nonlinear Neo-Hookean model of equation (5) is used with $E = 1 \times 10^6 \text{ N/mm}^2$, and $\nu = 0.2$. In computational contact mechanics using finite elements a search of the contacting facets and nodes is necessary. Mostly, the resultant normal vectors at the outer surfaces of the bodies are used to find the closest projection point on the other surface, see Yang et al. [2005] or Weißenfels and Wriggers [2015]. In these cases, the resultant nodal normal vectors are computed based on the facets of each element. However, using the divergence theorem based approach presented here the resultant normal vectors $n_S$ of equation (22) at each node $x_S$ are directly available and can be used to find the closest distance point $\bar{x}_S$ at the other surface, see also figure 8

$$\bar{x}_S = x_S - (x_S - \bar{x}_S) \cdot n_S = 0.$$ (27)
The subscript $S$ indicates a node at the surface of the block which is named slave surface in analogy to
the master-slave concept in computational contact mechanics. A node is in contact, if the penetration $g_N$ is larger than zero

$$g_{NS} = (x_S - \bar{x}_S) \cdot \frac{n_S}{\|n_S\|}, \quad g_{NS} \geq 0 \quad \text{active node}$$

$$g_{NS} < 0 \quad \text{inactive node} \quad (28)$$

In this approach, the penalty method is used to enforce the non penetration condition to hold. In this
case, the forces at active nodes can be directly approximated by using a penalty parameter

$$f_S = -c_N g_{NS} n_S. \quad (29)$$

The penalty parameter is chosen as $c_N = 10^8$ N/mm$^3$ allowing only small penetrations. As can be seen
in figure 9, the normal vectors automatically adjust during the calculation and updates the status of
the nodes at the slave surface following equation (28). Like mentioned in Yang et al. [2005] for the
Mortar method, for special cases a special treatment is necessary in computational contact mechanics.
If the contacting surfaces are flat, the influences of the adjacent surfaces at a vertex or at a edge are
normally neglected. This can be done in the same way using this approach. If the node-to-surface
concept is applied, the resultant nodal normal vectors are also advantageous to find the closest surface
to the corresponding node. Details on computational contact mechanics using the FEM can be found in
Wriggers [2006] and more recent developments in Lorenzis et al. [2017].

4.2 Compression of a structure with a gas filled hole

The next test case investigates the influence of a gas filled hole on the stress distribution and on the
deformation of a structure. An elastic block with a Young’s modulus of $E = 15$ N/cm$^2$ and a Poisson
ratio of $\nu = 0.3$ is loaded by a displacement of $w = 2$ cm on top of the block. The geometry, the dimensions and the boundary conditions are given in figure 10. Within this example the location and
the size of the hole is not known in advance.

However, using the proposed approach the surface and the volume of the hole can be specified without
any further information. Additionally, the load can be directly imposed. The nodes at the surface of the
inner hole are characterized by the resultant nodal normal vectors.

First a search domain is specified. Here, a cube is chosen with a side length of 4.5 cm around the center of the specimen. If the normalized
norm of the nodal normal vector

$$\frac{\|n_I\|}{\|n_I\|_{max}} \geq \delta \quad (30)$$

within this domain is larger than a given tolerance $\delta$, the node is characterized as a boundary node of
the inner hole.

The volume of the hole can also be directly computed using the resultant nodal normal vectors defined
in equation (22). Based on the divergence theorem

$$\int_{\Omega_x} \text{div} \mathbf{x} \, dv = \oint_{\partial\Omega_x} \mathbf{x} \cdot \mathbf{n} \, da \quad (31)$$

the enclosed volume results in the sum over the resultant nodal normal vectors multiplied with the
position vector of the corresponding node

$$v = \frac{1}{3} \left( \sum_{l=1}^{N_{sur}} x_{I_l} \cdot n_{I_l} \right). \quad (32)$$
Therein, $N_{\text{sur}}$ denotes the total number of surface nodes which are computed by means of equation (30). Using the proposed approach the calculated initial volume using equation (32) is 4.119 cm$^3$ leading to a radius of the hole of 1 cm.

The physics of the enclosed gas is modeled using Boyle’s law. It states that for ideal gases under isothermal conditions the product of pressure and volume is constant at all times. Hence, the pressure $p$ within the enclosed volume can be computed by means of the change in the volume $v$ during the deformation

$$p = \frac{p_0 v_0}{v}.$$  

(33)

The value $p_0$ and $v_0$ indicates the initial pressure and the initial volume of the hole, respectively, see also Rumpel and Schweizerhof [2003]. The force which acts at each node at the surface of the hole computes directly from the pressure and the resultant nodal normal vector

$$f_I = -\frac{p_0 v_0}{v} n_I.$$  

(34)

The minus sign stems from the fact that the nodal normal vectors directs outwards at the surface. The linearization of the above equation leads to a symmetric contribution to the stiffness matrix. Hence, the conservativeness of the volume depending loading is guaranteed using the proposed approach. During the deformation process the pressure inside is changing in dependency of the volume of the hole, see also figure 10. The influence of the gas within the hole can be seen in figure 11. The pressure due to the gas tries to keep up the shape of the volume during the deformation. In case of no gas within the hole the volume decreases much faster during the loading process.

4.3 Hydrostatic loading of an elastic structure

The last example demonstrates the application of a follower load where the pressure distribution is not equal at any point of the surface. Therein, the influence of the nodal integration concept, which is naturally involved in this formulation, is investigated. A cuboid container is filled with a fluid and the resulting deformation of the wall is calculated. The geometry, the dimensions and the boundary conditions can be found in figure 12. The material is assumed to behave nonlinear elastic and the Neo-Hookean model of equation (5) is used with $E = 1000$ N/cm$^2$, and $\nu = 0.3$. The load on the elastic wall

Figure 10: (Left) Geometry, dimensions and boundary conditions of the elastic structure. (Right) Pressure inside the whole in dependency of the volume.
Figure 11: Vertical normal stress distribution within the elastic structure after a loading of $w = 1.6$ cm. (Left) Gas filled hole. (Right) Without gas pressure.

Figure 12: (Left) Geometry, dimensions and boundary conditions of the filling of an elastic box. (Right) Side view of the box defining the position of the fluid surface and the normal vector on this surface.
is determined by means of the specific gravity $\gamma$ of the fluid, the position vector of the fluid surface $\mathbf{x}_W$, the normal vector on the surface $\mathbf{w}$ and the corresponding position vector at the node of the elastic wall.

$$\mathbf{f}_I = -p_I \mathbf{n}_I, \quad p_I = \gamma (\mathbf{x}_W - \mathbf{x}_I) \cdot \mathbf{w}. \quad (35)$$

The specific gravity $\gamma = \rho g = 0.1 \text{ N/cm}^3$ depends on the density of the fluid $\rho$ and the acceleration due to gravity $g$. The vector $\mathbf{w}$ is assumed to be constant and pointing in vertical direction during the filling process. The position vector of the water surface $\mathbf{x}_W$ depends on the deformation of the elastic structure and is also influenced by the incompressibility of the fluid. The latter one results in the additional constraint that the volume of the fluid has to be constant. Using the divergence theorem given in the previous example, see equation (31), and considering the upper free surface of the fluid $A_s$ the volume can be computed alternatively as

$$v = \frac{1}{3} \left( \mathbf{x}_W \cdot \mathbf{w} A_s + \sum_{I=1}^{N^{wet}} \mathbf{x}_I \cdot \mathbf{n}_I \right) = \text{const.} \quad (36)$$

The area of the free surface can be determined from a projection of the wetted surface of the structure, see also Rumpel and Schweizerhof [2004]. This can additionally be computed using the resultant normal vectors

$$A_s = \sum_{I=1}^{N^{wet}} \mathbf{w} \cdot \mathbf{n}_I. \quad (37)$$

Hence, the vertical component of the position vector of the upper free surface in equation (35) can be directly calculated from the volume, the resultant normal vectors and the coordinates of the wetted nodes

$$\mathbf{x}_W \cdot \mathbf{w} = \frac{1}{A_s} \left( 3v - \sum_{I=1}^{N^{wet}} \mathbf{x}_I \cdot \mathbf{n}_I \right). \quad (38)$$

The number of nodes $N^{wet}$ which are wetted by the fluid are determined by the current position vector of the upper free surface. Using the proposed approach due to the nodal integration process the pressure

![Figure 13](image.png)

**Figure 13**: (Left) Actual pressure distribution along the vertical wall of the container and pressure distribution in the discretized case. (Right) Front view showing the influence area of each wetted node in case of linear shape functions within the elements.

is assumed to be constant within the influence area of each node. This leads to a cascaded distribution of
the pressure in the discretized case, see also figure 13. However, by refining the mesh the nodal pressures approximates the original linear distribution. This can be demonstrated by an evaluation of the initial

![Graph showing relationship between volume and degrees of freedom.](image)

Figure 14: (Left) Initial volume of the filled fluid in dependency of the mesh resolution. (Right) Displacements in x-direction of the container due to the hydrostatic fluid loading.

volume. In this example the initial height of the fluid level is given as 19 cm leading to a volume for the fluid of 1539 cm$^3$. As can be seen in figure 14 with a refinement of the mesh the initial volume converges to the exact value. Hydrostatic fluid loading is a conservative problem and like in the previous example, the linearization also leads to a symmetric contribution to the stiffness matrix.

5 Conclusions

In this study an alternative approach to impose surface loads is presented. Using a nodal wise assembly procedure, applying the integration constraint and the divergence theorem a normal vector can be computed for each node. If the integration constraint is fulfilled, this vector is equal to zero for inner nodes. At the boundary this vector is perpendicular to the surface and its length corresponds to the influence area of that node. Each force vector results simply from the multiplication of the resultant nodal normal vector with the surface load. Hence, surface loads can be imposed directly based on volume element information only. The main advantage of this approach compared to the standard discretization scheme used within the Finite Element Method is that the resultant normal vectors are directly computed. Especially in contact cases where the surface can possess some kinks no special treatment of the normal vectors is necessary using this approach. Additionally, the imposition of follower loads including the linearization is straightforward. Boundary nodes can also be directly identified only from quantities of the volume and no surface parametrization is necessary. Although the applicability of the proposed approach is only demonstrated within the finite element framework, due to its generic derivation it can be applied within any Galerkin solution scheme. However, the only prerequisite is the fulfillment of the integration constraint.
References


